THE CONTINUOUS SHEARLET TRANSFORM IN HIGHER DIMENSIONS: VARIATIONS OF A THEME

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Abstract. This note is concerned with the generalization of the continuous shearlet transform to higher dimensions. Quite recently, a first approach has been derived in [4]. We present an alternative version which deviates from [4] mainly by a different generalization of the shear component. It turns out that the resulting integral transform is again associated with a square-integrable group representation.

1. Introduction

Modern technology allows for easy creation, transmission and storage of huge amounts of data. Confronted with a flood of data, such as internet traffic, or audio and video applications, nowadays the key problem is to extract the relevant information from these sets. To this end, usually the first step is to decompose the signal with respect to suitable building blocks which are well–suited for the specific application and allow a fast and efficient extraction. In this context, one particular problem which is currently in the center of interest is the analysis of *directional* information. In recent studies, several approaches have been suggested such as ridgelets [1], curvelets [2], contourlets [5], shearlets [11] and many others. For a general approach see also [10]. Among all these approaches, the shearlet transform stands out because it is related to group theory, i.e., this transform can be derived from a square-integrable representation $\pi : \mathcal{S} \to \mathcal{U}(L_2(\mathbb{R}^2))$ of a certain group $\mathcal{S}$, the so-called *shearlet group*, see [3]. Therefore, in the context of the shearlet transform, all the powerful tools of group representation theory can be exploited.

So far, the shearlet transform is well developed for problems in $\mathbb{R}^2$. However, for analyzing higher-dimensional data, there is clearly an urgent need for further generalization. In [4], a first approach in this direction has been presented. Similar to the two–dimensional case, the approach outlined there is based on translations, anisotropic dilations and specific shear matrices. It has been shown that the associated integral transform originates from a square-integrable representation of a group, the full n-variate shearlet group. Moreover, a very useful link to the important coorbit space theory developed by Feichtinger and Gröchenig [6, 7, 8] has been established and the potential to detect singularities has been demonstrated. In this note, we want to present a slightly different approach. It deviates from [4] mainly by the choice of the shear component. Instead of the block form used in [4], we work here with a suitable subgroup of Toeplitz matrices of the group of upper triangular matrices. Moreover, in contrary to the anisotropic (parabolic) dilation employed in [4], we restrict ourselves to isotropic dilations here. This setting yields a different generalization of the continuous shearlet transform. Nevertheless, similar to [4], the associated integral transform stems

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from a square-integrable group representation of a specific group, so that again all the powerful tools of group representation theory can be used.

This note is organized as follows. In Section 1, we establish our new underlying group and compute the associated Haar measures. Then, in Section 2, we show that this group indeed possesses a strictly continuous representation in $L_2(\mathbb{R}^n)$ which is moreover square integrable.

2. The Group Structure

In this section, we introduce a new version of the shearlet transform on $L_2(\mathbb{R}^n)$. This requires the generalization of the shear matrix. For $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}^{n-1}$, we set

$$A_a = \begin{pmatrix} a & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a \end{pmatrix} = aI_n \quad \text{and} \quad S_s = \begin{pmatrix} 1 & s_1 & s_2 & \dots & s_{n-1} \\ 0 & 1 & s_1 & s_2 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & s_1 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Lemma 2.1. *The set $\mathbb{R}^* \times \mathbb{R}^{n-1} \times \mathbb{R}^n$ endowed with the operation*

$$(a, s, t) \circ (a', s', t') = (aa', [S_{s'}^T S_s^T]_1, t + A_a S_s t') \ ,$$

where the bracket operation $[\cdot]_1$ extracts the last $n-1$ elements of the first column, is a locally compact group $\mathbb{S}$. The left and right Haar measures on $\mathbb{S}$ are given by

$$d\mu_l(a, s, t) = \frac{1}{|a|^{n+1}}\, da\, ds\, dt \quad \text{and} \quad d\mu_r(a, s, t) = \frac{1}{|a|}\, da\, ds\, dt.$$

Proof. A direct computation shows that $e := (1, 0, 0)$ is the neutral element in $\mathbb{S}$. Let us denote $S_s^{-T} = (S_s^{-1})^T$ and observe that $S_{[S_s^T]_1} = S_s$. The inverse of $(a, s, t) \in \mathbb{R}^* \times \mathbb{R}^{n-1} \times \mathbb{R}^n$ is given by

$$(a, s, t)^{-1} = \left(\frac{1}{a}, [S_s^{-T}]_1, -S_s^{-1} A_{\frac{1}{a}} t \right),$$

since

$$\begin{aligned} (a, s, t) \circ \left(\frac{1}{a}, [S_s^{-T}]_1, -S_s^{-1} A_{\frac{1}{a}} t \right) &= \left(\frac{1}{a} a, [S_s^{-T} S_s^T]_1, -A_a S_s S_s^{-1} A_{\frac{1}{a}} t + t \right) \\ &= (1, 0, 0). \end{aligned}$$

Note that by induction arguments one easily verifies that S_s^{-1} is again of Töplitz type as S_s. Furthermore, the multiplication is associative. With the observation that for $s, r \in \mathbb{R}^{n-1}$ and $a \in \mathbb{R}^*$

$$\left(S_{[S_s^T S_r^T]_1} \right)^T = S_s^T S_r^T \text{ and } A_a S_s = S_s A_a,$$

we have

$$\begin{aligned}((a,s,t)\circ(a',s',t'))\circ(\tilde a,\tilde s,\tilde t) &= (aa',[S_{s'}^TS_s^T]_1,t+A_aS_st')\circ(\tilde a,\tilde s,\tilde t)\\ &= (aa'\tilde a,[S_{\tilde s}^TS_{s'}^TS_s^T]_1,S_{[S_{s'}^TS_s^T]_1}A_{aa'}\tilde t+t+A_aS_st')\\ &= (aa'\tilde a,[S_{\tilde s}^TS_{s'}^TS_s^T]_1,S_sS_{s'}A_aA_{a'}\tilde t+A_aS_st'+t)\\ &= (aa'\tilde a,[S_{\tilde s}^TS_{s'}^TS_s^T]_1,A_aS_s(A_{a'}S_{s'}\tilde t+t')+t)\\ &= (a,s,t)\circ(a'\tilde a,[S_{\tilde s}^TS_{s'}^T]_1,A_{a'}S_{s'}\tilde t+t')\\ &= (a,s,t)\circ((a',s',t')\circ(\tilde a,\tilde s,\tilde t)).\end{aligned}$$

It remains to compute the Haar measures. Observing that the vector $[S_s^TS_{s'}^T]_1$ can be seen as a translation of s, we have for a function F on $\mathbb{S}$

$$\begin{aligned}&\int_{\mathbb{R}^*}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n}F(a'a,[S_s^TS_{s'}^T]_1,t'+A_{a'}S_{s'}t)\,dt\,ds\,\frac{da}{|a|^{n+1}}\\ &=\int_{\mathbb{R}^*}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n}F(a'a,[S_s^TS_{s'}^T]_1,t)\,\frac{dt}{|\det(A_{a'}S_{s'})|}\,ds\,\frac{da}{|a|^{n+1}}\\ &=\int_{\mathbb{R}^*}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n}F(a'a,s,t)\,dt\,ds\,\frac{da}{|a'|^n|a|^{n+1}}\\ &=\int_{\mathbb{R}^*}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n}F(\tilde a,s,t)\,dt\,ds\,\frac{d\tilde a}{|a'|^n|\frac{\tilde a}{a'}|^{n+1}|a'|}\\ &=\int_{\mathbb{R}^*}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n}F(\tilde a,s,t)\,dt\,ds\,\frac{d\tilde a}{|\tilde a|^{n+1}}\,,\end{aligned}$$

so that $d\mu_l$ is indeed the left Haar measure on $\mathbb{S}$. Similarly we can verify that $d\mu_r$ is the right Haar measure on $\mathbb{S}$. □

In the following, we will use only the left Haar measure and use the abbreviation $d\mu = d\mu_l$.

3. The Representation

For $f\in L_2(\mathbb{R}^n)$ we define for $(a,s,t)\in\mathbb{S}$

$$(\pi(a,s,t)f)(x) = f_{a,s,t}(x) =: |a|^{-n/2}f(A_a^{-1}S_s^{-1}(x-t)). \tag{1}$$

It is easy to check that $\pi:\mathbb{S}\to\mathcal{U}(L_2(\mathbb{R}^n))$ is a mapping from $\mathbb{S}$ into the group $\mathcal{U}(L_2(\mathbb{R}^n))$ of unitary operators on $L_2(\mathbb{R}^n)$. The *Fourier transform* of $f_{a,s,t}$ is given by

$$\begin{aligned}(\hat\pi(a,s,t)\hat f)(\omega) &= |a|^{-n/2}\left(f(A_a^{-1}S_s^{-1}(\cdot-t))\right)^\wedge(\omega)\\ &= |a|^{-n/2}|\det(A_a^{-1}S_s^{-1})|^{-1}\hat f(S_s^TA_a^T\omega)e^{-2\pi i\langle t,\omega\rangle}\\ &= |a|^{n/2}\hat f(S_s^TA_a^T\omega)e^{-2\pi i\langle t,\omega\rangle}.\end{aligned}\tag{2}$$

Recall that a *unitary representation* of a locally compact group G with the left Haar measure μ on a Hilbert space $\mathcal{H}$ is a homomorphism π from G into the group of unitary operators $\mathcal{U}(\mathcal{H})$ on $\mathcal{H}$ which is continuous with respect to the strong operator topology.

Lemma 3.1. *The mapping π defined by* (1) *is a unitary representation of* $\mathbb{S}$.

Proof. Note that the representations π and $\hat{\pi}$ are equivalent. Let $\psi \in L_2(\mathbb{R}^n)$, $\omega \in \mathbb{R}^n$, and (a,s,t), $(a',s',t') \in \mathbb{S}$. Therefore, we obtain

$$\begin{aligned}
\hat{\pi}(a,s,t) \circ \hat{\pi}(a',s',t')\psi(\omega) &= \hat{\pi}(a,s,t)\left(\hat{\pi}(a',s',t')\psi\right)(\omega) \\
&= |a|^{n/2}\left(\hat{\pi}(a',s',t')\psi\right)(S_s^T A_a^T \omega) e^{-2\pi i \langle t,\omega\rangle} \\
&= |a|^{n/2}|a'|^{n/2}\psi(S_{s'}^T A_{a'}^T S_s^T A_a^T \omega) e^{-2\pi i \langle t', S_s^T A_a^T \omega\rangle} e^{-2\pi i \langle t,\omega\rangle} \\
&= |aa'|^{n/2}\psi(S_{s'}^T S_s^T A_{aa'}^T \omega) e^{-2\pi i \langle A_a S_s t' + t,\omega\rangle} \\
&= \hat{\pi}\left((a,s,t)\circ(a',s',t')\right)\psi(\omega).
\end{aligned}$$

□

A nontrivial function $\psi \in L_2(\mathbb{R}^n)$ is called *admissible*, if

$$0 < \int_{\mathbb{S}} |\langle \psi, \pi(a,s,t)\psi\rangle|^2 d\mu(a,s,t) < \infty.$$

If π is irreducible and there exits at least one admissible function $\psi \in L_2(\mathbb{R}^3)$, then π is called *square integrable*. The following result shows that the unitary representation π defined in (1) is square integrable.

Theorem 3.2. *A function $\psi \in L_2(\mathbb{R}^n)$ is admissible if and only if it fulfills the* admissibility condition

$$0 < C_\psi := \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\omega)|^2}{|\omega_1|^n}\, d\omega < \infty. \tag{3}$$

Then, for any $f \in L_2(\mathbb{R}^n)$, the following equality holds true:

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2\, d\mu(a,s,t) = C_\psi\, \|f\|^2_{L_2(\mathbb{R}^n)}\,. \tag{4}$$

In particular, the unitary representation π is irreducible and hence square integrable.

Proof. Observe that

$$\mathcal{SH}_f(a,u,t) := \langle f, \psi_{a,s,t}\rangle = \langle f, |a|^{-n/2}\psi(A_a^{-1}S_s^{-1}(\cdot - t))\rangle = f * \psi^*_{a,s,0}(t), \tag{5}$$

where $\psi^*_{a,s,t}(x) := \overline{\psi_{a,s,t}(-x)}$. Employing the Plancherel theorem, (2), and (5) we obtain

$$\begin{aligned}
\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2 \frac{da}{|a|^{n+1}}\, ds\, dt &= \int_{\mathbb{S}} |f * \psi^*_{a,s,0}(t)|^2 dt\, ds\, \frac{da}{|a|^{n+1}} \\
&= \int_{\mathbb{R}^*}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n} |\hat{f}(\omega)|^2 |\widehat{\psi^*}_{a,s,0}(\omega)|^2 d\omega\, ds\, \frac{da}{|a|^{n+1}} \\
&= \int_{\mathbb{R}^*}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n} |\hat{f}(\omega)|^2 |a|^{-1} |\hat{\psi}(S_s^T A_a^T \omega)|^2 d\omega\, ds\, da \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^*}\int_{\mathbb{R}^{n-1}} |\hat{f}(\omega)|^2 |a|^{-1} |\hat{\psi}(S_s^T A_a^T \omega)|^2\, ds\, da\, d\omega.
\end{aligned}$$

Observing that

$$S_s^T A_a^T \omega = \begin{pmatrix} 1 & 0 & & \dots & 0 \\ s_1 & 1 & 0 & & \\ s_2 & s_1 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ s_{n-1} & s_{n-2} & \dots & s_1 & 1 \end{pmatrix} \begin{pmatrix} a\omega_1 \\ a\omega_2 \\ \vdots \\ \vdots \\ a\omega_n \end{pmatrix} = \begin{pmatrix} a\omega_1 \\ s_1 a\omega_1 + a\omega_2 \\ \vdots \\ \vdots \\ s_{n-1}a\omega_1 + as_{n-2}\omega_2 + \ldots + a\omega_n \end{pmatrix},$$

we may subsequently substitute $\tilde{s}_{n-1} := s_{n-1}a\omega_1 + \ldots + a\omega_n$, $\tilde{s}_{n-2} := s_{n-2}a\omega_1 + \ldots + a\omega_{n-2}, \ldots$, $\tilde{s}_1 := s_1 a\omega_1 + a\omega_2$. This yields

$$\begin{aligned}\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2 \frac{da}{|a|^{n+1}}\, ds\, dt &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^*}\int_{\mathbb{R}^{n-1}} |\hat{f}(\omega)|^2 |a|^{-n} |\omega_1|^{-(n-1)} |\hat{\psi}(a\omega_1, \tilde{s}_1, \ldots, \tilde{s}_{n-1})|^2\, d\tilde{s}\, da\, d\omega \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^*}\int_{\mathbb{R}^{n-1}} |\hat{f}(\omega)|^2 |\tilde{a}|^{-n} |\hat{\psi}(\tilde{a}, \tilde{s}_1, \ldots, \tilde{s}_{n-1})|^2\, d\tilde{s}\, d\tilde{a}\, d\omega \\ &= \int_{\mathbb{R}^n} |\hat{f}(\omega)|^2 d\omega \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\omega)|^2}{|\omega_1|^n}\, d\omega.\end{aligned}$$

□

4. Retrieval of Geometric Information

In this section, we deal with the decay of the shearlet transform at hyperplane singularities. An $(n-m)$-dimensional hyperplane in $\mathbb{R}^n$, $1 \le m \le n$, not containing the x_1-axis can be written w.l.o.g. as

$$\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}}_{x_A} + P \underbrace{\begin{pmatrix} x_{m+1} \\ \vdots \\ x_n \end{pmatrix}}_{x_E} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad P := \begin{pmatrix} p_1^{\mathrm{T}} \\ \vdots \\ p_m^{\mathrm{T}} \end{pmatrix} \in \mathbb{R}^{m,n-m}.$$

A characterization of this hyperplane is given via the Delta distribution δ by $\nu_m := \delta(x_A + Px_E)$ with Fourier transform $\hat{\nu}_m(\omega) = \delta(\omega_E - P^{\mathrm{T}}\omega_A)$.

The following theorem describes the decay of the shearlet transform at hyperplane singularities. We use the notation $\mathcal{SH}_\psi f(a,s,t) \sim |a|^r$ as $a \to 0$, if there exist constants $0 < c \le C < \infty$ such that

$$c|a|^r \le \mathcal{SH}_\psi f(a,s,t) \le C|a|^r \text{ as } a \to 0.$$

Theorem 4.1. *Let $\psi \in L_2(\mathbb{R}^n)$ be a shearlet satisfying $\hat{\psi} \in C^\infty(\mathbb{R}^n)$. Assume that for some $k > 1$ the shearlet is defined by $\hat{\psi}(\omega) = \hat{\psi}_1(\omega_1)\hat{\psi}_2(\tilde{\omega}/\omega_1^k)$, where $\operatorname{supp}\hat{\psi}_1 \in [-a_1, -a_0] \cup [a_0, a_1]$ for some $a_1 > a_0 \ge \alpha > 0$ and $\operatorname{supp}\hat{\psi}_2 \in ([-b_1, -b_0] \cup [b_0, b_1])^{n-1}$ for $b_1 > b_0 \ge \beta > 0$. Furthermore, let $\tilde{S}$ denote the shear matrix generated by $(s_1, \ldots, s_{n-2})$ and let $\bar{s} = \tilde{S}^{-T}s$. If*

$$(\bar{s}_m, \ldots, \bar{s}_{n-1}) = (-1, \bar{s}_1, \ldots, \bar{s}_{m-1})\, P \quad \text{and} \quad (t_1, \ldots, t_m) = -(t_{m+1}, \ldots, t_n)\, P^T,$$

then

$$\mathcal{SH}_\psi \nu_m(a,s,t) \sim |a|^{\frac{n}{2}-m} \quad \text{as } a \to 0. \tag{6}$$

Otherwise, the shearlet transform $\mathcal{SH}_\psi \nu_m$ decays rapidly as $a \to 0$.

The support condition on $\hat{\psi}_1$ and $\hat{\psi}_2$ can be relaxed toward a rapid decay of the functions.

Proof. An application of Plancherel's theorem for tempered distribution yields

$$\begin{aligned}\mathcal{SH}_\psi \nu_m(a,s,t) &:= \langle \nu_m, \psi_{a,s,t}\rangle \\ &= \langle \hat{\nu}_m, \hat{\psi}_{a,s,t}\rangle \\ &= |a|^{\frac{n}{2}} \int_{\mathbb{R}^m} e^{2\pi i\langle t_A + Pt_E, \omega_A\rangle} \bar{\hat{\psi}} \left(S_s^T A_a^T \begin{pmatrix} \overline{\omega}_A \\ P^{\mathrm{T}}\omega_A \end{pmatrix}\right) d\omega_A\end{aligned}$$

with $\overline{\omega}_A = (\omega_2, \ldots, \omega_m)^{\mathrm{T}}$. Rewriting

$$S_s^T A_a^T \begin{pmatrix} \overline{\omega}_A \\ P^{\mathrm{T}}\omega_A \end{pmatrix} = \begin{pmatrix} a\omega_1 \\ a\omega_1 \begin{pmatrix} s_1 \\ \vdots \\ s_{n-1}\end{pmatrix} + a\tilde{S}^T \begin{pmatrix} \overline{\omega}_A \\ P^{\mathrm{T}}\omega_A \end{pmatrix}\end{pmatrix},$$

where $\tilde{S}$ is the $(n-1)\times(n-1)$ submatrix of S_s, which is also a shear matrix generated by $(s_1, \ldots, s_{n-2})$, it follows by the definition of $\hat{\psi}$ that

$$\mathcal{SH}_\psi \nu_m(a,s,t) = |a|^{\frac{n}{2}} \int_{\mathbb{R}^m} e^{2\pi i\langle t_A + Pt_E, \omega_A\rangle} \bar{\hat{\psi}}_1(a\omega_1) \bar{\hat{\psi}}_2 \left(a^{1-k}(\omega_1^{1-k} s + \omega_1^{-k}\tilde{S}^T \begin{pmatrix} \overline{\omega}_A \\ P^{\mathrm{T}}\omega_A \end{pmatrix})\right) d\omega_A.$$

Substituting $\tilde{\xi}_A = (\xi_2, \ldots, \xi_m)^{\mathrm{T}} := \overline{\omega}_A/\omega_1^k$, i.e., $d\tilde{\omega}_A = |\omega_1|^{k(m-1)}\, d\tilde{\xi}_A$, we get

$$\begin{aligned}\mathcal{SH}_\psi \nu_m(a,s,t) &= |a|^{\frac{n}{2}} \int_{\mathbb{R}^m} e^{2\pi i\langle t_A + Pt_E, (\omega_1, \omega_1^k \tilde{\xi}_A^{\mathrm{T}})^{\mathrm{T}}\rangle} \bar{\hat{\psi}}_1(a\omega_1)|\omega_1|^{k(m-1)} \\ &\quad \times \bar{\hat{\psi}}_2 \left(\tilde{S}^T (a^{1-k}\omega_1^{1-k}\tilde{S}^{-T}s + a^{1-k}\begin{pmatrix} \tilde{\xi}_A \\ P^{\mathrm{T}}(\omega_1^{1-k}, \tilde{\xi}_A^{\mathrm{T}})^{\mathrm{T}}\end{pmatrix})\right) d\tilde{\xi}_A d\omega_1\end{aligned}$$

and by setting $\bar{s} = \tilde{S}^{-T}s$, $\bar{s}_a = (\bar{s}_1, \ldots, \bar{s}_{m-1})^{\mathrm{T}}$, and $\bar{s}_e = (\bar{s}_m, \ldots, \bar{s}_{n-1})^{\mathrm{T}}$ we obtain by substituting $\tilde{\omega}_A := a^{1-k}(\omega_1^{1-k}\bar{s}_a + \tilde{\xi}_A)$

$$\begin{aligned}\mathcal{SH}_\psi \nu_m(a,s,t) &= |a|^{\frac{n}{2}+(m-1)(k-1)} \int_{\mathbb{R}^m} e^{2\pi i\langle t_A + Pt_E, (\omega_1, (\omega_1^k/a^{1-k}\tilde{\omega}_A - \omega_1 \bar{s}_a)^{\mathrm{T}})^{\mathrm{T}}\rangle} \bar{\hat{\psi}}_1(a\omega_1)|\omega_1|^{k(m-1)} \\ &\quad \times \bar{\hat{\psi}}_2 \left(\tilde{S}^{\mathrm{T}} \begin{pmatrix} \tilde{\omega}_A \\ a^{1-k}\omega_1^{1-k}(\bar{s}_e + P^{\mathrm{T}}\begin{pmatrix} 1 \\ -\bar{s}_a\end{pmatrix}) + P^{\mathrm{T}}\begin{pmatrix} 0 \\ \tilde{\omega}_A\end{pmatrix}\end{pmatrix}\right) d\tilde{\omega}_A d\omega_1\end{aligned}$$

If the vector

$$\bar{s}_e - P^{\mathrm{T}}\begin{pmatrix} -1 \\ \bar{s}_a \end{pmatrix} \neq 0_{n-m} \tag{7}$$

then at least one component of its product with a^{1-k} becomes arbitrary large as $a \to 0$. By the support property of $\hat{\psi}_2$, we conclude that $\hat{\psi}_2(\tilde{S}^{\mathrm{T}}(\tilde{\omega}_A, \cdot)^{\mathrm{T}})$ becomes zero if $\tilde{\omega}_A$ is not in $([-b_1, -b_0] \cup [b_0, b_1])^{m-1} \subset R^{m-1}$. But for all $\tilde{\omega}_A \in ([-b_1, -b_0] \cup [b_0, b_1])^{m-1}$ at least one component of

$$a^{1-k}\omega_1^{1-k}(\bar{s}_e - P^{\mathrm{T}}\begin{pmatrix} -1 \\ \bar{s}_a\end{pmatrix}) + P^{\mathrm{T}}\begin{pmatrix} 0 \\ \tilde{\omega}_A\end{pmatrix}$$

is not within the support of $\hat{\psi}_2$ for a sufficiently small so that $\hat{\psi}_2$ becomes zero again. Assume now that we have equality in (7). Then

$$\begin{aligned}\mathcal{SH}_\psi \nu_m(a,s,t) &= |a|^{\frac{n}{2}+(m-1)(k-1)} \int_{\mathbb{R}^m} e^{2\pi i \langle t_A + Pt_E, (\omega_1, (\omega_1^k/a^{1-k}\tilde{\omega}_A - \omega_1 \bar{s}_a)^{\mathrm{T}})^{\mathrm{T}}\rangle} \bar{\hat{\psi}}_1(a\omega_1)|\omega_1|^{k(m-1)} \\ &\quad \times \bar{\hat{\psi}}_2\left(\tilde{S}^{\mathrm{T}}\begin{pmatrix}\tilde{\omega}_A \\ P^{\mathrm{T}}\begin{pmatrix}0\\ \tilde{\omega}_A\end{pmatrix}\end{pmatrix}\right) d\tilde{\omega}_A d\omega_1 \\ &= |a|^{\frac{n}{2}-m} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} e^{-2\pi i \frac{\xi_1}{a}\langle t_A + Pt_E, (1, \xi_1^{k-1}\tilde{\omega}_A^{\mathrm{T}} - \bar{s}_a^{\mathrm{T}})^{\mathrm{T}}\rangle} \bar{\hat{\psi}}_1(\xi_1)|\xi_1|^{k(m-1)} d\xi_1 \\ &\quad \times \bar{\hat{\psi}}_2\left(\tilde{S}^{\mathrm{T}}\begin{pmatrix}\tilde{\omega}_A \\ P^{\mathrm{T}}\begin{pmatrix}0\\ \tilde{\omega}_A\end{pmatrix}\end{pmatrix}\right) d\tilde{\omega}_A\ .\end{aligned}$$

Consequently, for $t_A + Pt_E = 0$, we have

$$\mathcal{SH}_\psi \nu_m(a,s,t) = |a|^{\frac{n}{2}-m} \int_{\mathbb{R}} \bar{\hat{\psi}}_1(\xi_1)|\xi_1|^{k(m-1)} d\xi_1 \int_{\mathbb{R}^{m-1}} \bar{\hat{\psi}}_2\left(\tilde{S}^{\mathrm{T}}\begin{pmatrix}\tilde{\omega}_A \\ P^{\mathrm{T}}\begin{pmatrix}0\\ \tilde{\omega}_A\end{pmatrix}\end{pmatrix}\right) d\tilde{\omega}_A \sim |a|^{\frac{n}{2}-m}\ .$$

For $t_A + Pt_E \neq 0$, consider $\tilde{\psi}_{a,\tilde{\omega}_A}(\xi_1) := e^{2\pi i \langle t_A + Pt_E, (\xi_1, \xi_1^k \tilde{\omega}_A^{\mathrm{T}})^{\mathrm{T}}\rangle / a} \bar{\hat{\psi}}_1(\xi_1)|\xi_1|^{k(m-1)}$ which is for all $\tilde{\omega}_A$ and a again in C^∞. Due to the support property of $\hat{\psi}_2$, the integration with respect to $\tilde{\omega}_A$ is over some finite domain. Therefore, $\lim_{a\to 0} \hat{\tilde{\psi}}_{a,\tilde{\omega}_A}(\langle t_A + Pt_E, (-1, \bar{s}_a^{\mathrm{T}})^{\mathrm{T}}\rangle / a)$ is uniformly in $\tilde{\omega}_A$. Hence, limes and integration can be exchanged and since $\hat{\tilde{\psi}}_{a,\tilde{\omega}_A}$ is a rapidly decaying function, the limes for $a \to 0$ is zero implying that $\mathcal{SH}_\psi \nu_m(a,s,t)$ decays rapidly as well. (Since the parameter a in the definition of $\tilde{\psi}_{a,\tilde{\omega}_A}$ only appears in the exponential term, general results on Fourier transform imply that the dependency on a does not effect these arguments, see, e.g., [9], proof of Corollary 8.23 for details). □

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