Continuous wavelet methods in signal processing

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1 Introduction

Some of the most complex applications in signal and image processing, e.g. surveying production processes or analyzing meteorological data (air pressure, wind speed, etc.), require online, quantitative estimates of relevant system parameters.

The nature of wavelet methods for these applications has changed considerably over the last five years. Earlier applications were mainly concerned with choosing an appropriate wavelet and computing as well as visualizing the related transforms. These methods were usually based on the fast discrete wavelet transform, they already proved to be successful for a large variety of basic problems in signal and image processing.

Since then the focus has shifted to more complex applications and to the development of adapted tools for an online quantitative evaluation of the resulting wavelet transforms. Moreover, the complexity of these applications often require to compute a full continuous wavelet transform, which requires different algorithms and substantially more computer power compared to the fast discrete transforms. Hence, we will concentrate on recent results concerning continuous wavelet transforms in this short review paper.

This paper starts with a short comparison between Fourier and wavelet transforms and a sketch of some basic properties of wavelet methods. The second chapter describes some recent theoretical developments for continuous wavelet transforms, these results are illustrated by two prototypical applications.

2 Continuous wavelet transforms

Fourier methods still provide the most frequently used tools in industrial signal processing. The Fourier transform correlates a given signal $f(t)$ with pure frequencies

$$\hat{f}(\omega) \;=\; \frac{1}{\sqrt{2\pi}} \int\limits_R f(t)\; e^{-it\omega}\; dt \quad .$$

The maxima of the Fourier spectrum e.g. reveal dominant frequencies in f, i.e. they allow to analyze stationary, periodic or quasi-periodic structures.

However, due to the global support of pure frequencies, the classical Fourier transform fails when investigating local structures like instationary, transient processes. This was the motivation for introducing windowed Fourier transforms,

$$\mathcal{G}f(\omega, b) \;=\; \frac{1}{\sqrt{2\pi}} \int\limits_{R} f(t)\; \varphi(t-b)\; e^{-it\omega}\; dt \quad .$$

The multiplication with a window function φ first localizes the data around $t = b$ before computing its Fourier transform. This procedure yields a time–frequency decomposition of f. Nevertheless, the ability to analyze local structures is limited by the support of φ, which has to be fixed a priori.

The continuous wavelet transform was introduced in the early 80's in order to overcome this limited time–frequency localization of Fourier methods for non–stationary signals. The wavelet transform correlates the signal f with a shifted and translated test function ψ, $(a, b \in I\!R, a \neq 0)$:

$$L_\psi f(a,b) \;=\; |a|^{-1/2} \int\limits_{I\!R} f(t)\psi\left(\frac{t-b}{a}\right)\; dt \quad .$$

This transforms analyzes local features of f at time $t = b$. The parameter a determines the scale (or size of details) which is examined, the scale becomes finer and finer as a approaches 0.

Many papers have been written on differences and similarities between Fourier and wavelet transforms, for more detailed information see e.g. [1,2,3]. A pair of functions $(\psi, \tilde{\psi})$ is called a wavelet pair, if it satifies the admissibility condition, ($\hat{\psi}$ denotes the Fourier transform of ψ):

$$0 < \int\limits_{I\!R} \frac{|\hat{\psi}(\omega)\hat{\tilde{\psi}}(\omega)|}{|\omega|}\; d\omega < \infty \quad .$$

This condition insures that f can be recovered from its wavelet transform $L_\psi f$ by

$$f(t) \;=\; \int\limits_{I\!R}\int\limits_{I\!R} L_\psi f(a,b)\; |a|^{-1/2}\; \tilde{\psi}\left(\frac{t-b}{a}\right)\; \frac{da db}{a^2} \quad .$$

Some of the most interesting cases arise when $\psi = \tilde{\psi}$, in which case ψ is called a wavelet. The admissibility condition then essentially states that ψ has zero

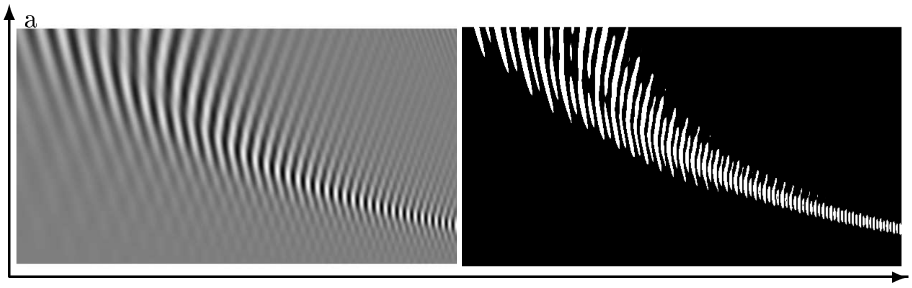

Figure 1: The mexican hat was used to compute the wavelet transform of the chirp signal. The plot on the right shows the extrema of the wavelet transform, they show that the size of details decreases continuously.

mean:

$$\int_{\mathbb{R}} \psi(t)\ dt = 0 \quad .$$

Standard examples for 1d-wavelets are:

- Haar wavelet: $\psi(t) = \begin{cases} 0 & \text{for } \ t < 0 \text{ or } t > 1 \\ 1 & \text{for } \ 0 \leq t < 1/2 \\ -1 & \text{for } \ 1/2 \leq t \leq 1 \end{cases}$,
- Mexican hat wavelet: $\psi(t) = -\frac{d^2}{dt^2} e^{-t^2/2} = (1-t^2)e^{-t^2/2}$,
- Morlet wavelet: $\psi(t) = \pi^{-1/4} \left(e^{-i\ell t} - e^{-\ell^2/2} \right) e^{-t^2/2}$, this wavelet is parameterized by ℓ whose optimal choice depends on the application.

Figure 1 depicts the wavelet transform of a chirp–signal

$$f(t) = sin\left(t^{5/2}\right) \quad .$$

This is a typical signal which is difficult to analyse with Fourier or windowed Fourier techniques since its local frequency increases polynomially, or equivalently the size of details decreases. It therefore exhibits a truly multiscale structure. Figure 1 also displays its wavelet transform in the upper (b, a)–plane after thresholding, the local extrema clearly reveal the decreasing size of details.

Most applications in signal processing require more than a visual inspection of the wavelet transform. Therefore we list some of the most important tools for interpretating the wavelet transform. We always start from a mathematical property of $L_\psi f(a,b)$ followed by a description of its impact in signal processing.

2.1 Asymptotic behaviour for $a \to 0$

Let us consider the Haar wavelet and its wavelet transform. $L_\psi f(a,b)$ then denotes the (weighted) difference between the average of f over the interval $[b, b+a/2)$ and the average of f over the interval $[b+a/2, b+a)$. As $a \to 0$ this approximates the first derivative of f at $t = b$.
More general one can define the "order of a wavelet", [4], depending on the asymptotic behaviour of $L_\psi f(a,b)$ as $a \to 0$. E.g. the Haar wavelet is of order 1, the mexican hat wavelet is of order 2.

Lemma *If ψ is a wavelet of order k and if f is k–times differentiable at $t = b$, then*

$$|a|^{-k-1/2}\ L_\psi f(a,b) \longrightarrow c\ f^{(k)}(b) \quad as \quad a \to 0 \quad .$$

The asymptotic behaviour of the wavelet transform therefore allows a local regularity estimate for f. This pointwise regularity analysis could not be carried out with the Fourier transform, since its asymptotic behaviour as $\omega \to \infty$ only yields a global regularity estimate.
In signal processing this property can be used to detect and localize discontinuities (jumps, cracks, defects): $L_\psi f(a,b)$ decays to zero at regular points $t = b$ but diverges at points where f is not sufficiently differentiable.
More precisely: depending on the mathematical model for the discontinuity one may choose an appropriate wavelet. If e.g. one searches for hidden jumps – they relate to points where f is not differentiable – then an analysis with the Haar wavelet is sufficient to detect those irregularities.
Figure 2 displays the wavelet transform of a smooth function $f(t) = e^{-t^2/2}$ with a small defect at time $t = 0.2$. The Haar wavelet was used in this example.

2.2 Wavelet extrema

The most obvious interpretation of the wavelet transform is based on the local extrema of $L_\psi f(a,b)$.
Let us consider the problem of detecting a pattern whose shape is similar to ψ but whose size is unknown. In this case a pattern matching approach with varying pattern size is needed.
The wavelet transform does exactly that: it correlates the signal f with shifted and scaled version of the pattern ψ, hence a local extrema of $L_\psi f(a,b)$ reveals

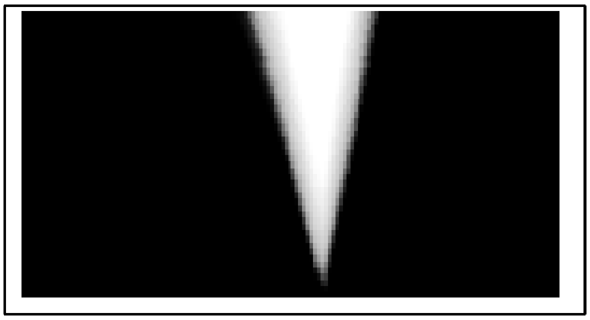

Figure 2: Choosing the Haar wavelet and displaying its wavelet transform reveals discontinuities of the signal. In this case a smooth signal with a small defect at $t = 0.1$ was analyzed. The plot displays the wavelet transform for $b \in [-1, 1]$, $a \in [0, 0.5]$.

a searched for pattern of size a at time $t = b$.
Note that – since wavelets have zero mean – this analysis is independent of shifts by constant. Moreover if ψ has more vanishing moments then the computation is independent of linear or polynomial trends in the data f.

2.3 Indicator function

Suppose that the scale (size) of a searched for pattern is unknown in some application. The scale – energy spectrum or indicator function

$$n(a) \;=\; \int_{\mathbb{R}} |L_\psi f(a,b)|^2 \; db$$

sums the L^2–energy of all details on a specified scale. If the energy is evenly distributed over all scales, then $n(a)$ is expected to exhibit a regular decay as $a \to 0$. In any case a deviation or local maxima of $n(a)$ reveals that f has some significant structure on the scale a.
For some classes of functions more specific power laws can be proved. E.g. turbulent signals have decay rates

$$n(a) \sim a^{\kappa},$$

where κ varies between 5/3 (fully developed turbulence) and 3 (non turbulent flows). This can be used to classify turbulence measurements, see e.g. [5], where wind measurements were evaluated in order to detect cirrus clouds. Similar experiments are presently carried out by D. Handorf, Deutscher Wetterdienst, Potsdam, who is examining antartical wind measurements.

2.4 Wavelets and bandpass filtering

For a fixed value of a one can view the wavelet transform as a convolution of f with a scaled wavelet ψ. The convolution property of the Fourier transform gives the following result.

Lemma: *Assume that $\psi,\ f \in L^2(I\!R)$. The one–dimensional Fourier transform of $L_\psi f(a,b)$ with respect to the variable b yields*

$$\widehat{L_\psi f}(a,\omega) \;=\; c|a|^{1/2}\hat{f}(\omega)\hat{\psi}(a\omega) \ .$$

For fixed a the wavelet transform of f can thus be viewed as a bandpass filter with filter function $\psi(t/a)$. Since the admissibility condition forces the Fourier transform of ψ to vanish at $\omega = 0$ this is indeed a bandpass filter with a variing frequency band depending on a. The wavelet transform therefore simultaneously computes bandpass filters for a complete family of frequency bands.
For a more detailed description see [2].

3 Mathematical research directions for continuous wavelet transforms

The mathematical research in this field presently centers around two basic questions, namely the construction of wavelets with prescribed properties and the quantitative evaluation of wavelet transforms.

3.1 Construction principles for multidimensional wavelets

The construction of wavelets for continuous transforms differs completely from construction principles for discrete wavelets. In this chapter we will concentrate on recent results concerning multidimensional, anisotropic wavelets for continuous transforms.

For higher dimensional transforms one has to replace the scaling factor a in (1) by an invertible scaling matrix A. This choice is too general: in order to obtain an isometric transform, which e.g. ensures that the adjoint transform constitutes the inverse transformation, one commonly restricts A to a two-parameter family of matrices, see [6,7]

$$L_\psi f(a,\theta,b) \;=\; |a|^{-n/2} \int\limits_{I\!R^n} f(t)\;\psi\left(\frac{1}{a}R(\theta)^{-1}(x-b)\right)\;dx \quad ,$$

where $b \in I\!R^n, a \in I\!R$ and $R(\theta^)$ denotes a unitary rotation matrix. In the two dimensional case

$$R(\theta) = \left(\begin{array}{cc} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{array}\right) \;,$$

with $\theta \in [0, 2\pi)$.

Possible choices for ψ include straight forward extensions of the 1d-wavelets, e.g.

- Mexican hat wavelet: $\psi(x) = (1-|x|^2)e^{-|x|^2/2}$,
- isotropic Morlet wavelet: $\psi(x) = \pi^{-1/4}\;\left(e^{-i\ell|x|} - e^{-\ell^2/2}\right)\;e^{-|x|^2/2}$.

However the related transform a rotational invariant, they exhibit a poor angular resolution. The analysis of directed structures was enhanced by the introduction of the anisotropic Morlet wavelet [6]

$$\psi(x) = \exp(ik_0 \cdot x) \cdot \exp(-\frac{1}{2}|Ax|^2) + \text{corr.term}\;,$$

where $k_0 \in I\!R^n$ and $A = \text{diag}(\epsilon_1^2, \ldots, \epsilon_n^2)$ with $\epsilon_i^2 \geq 1$, which has been widely used for various applications, see [6,8,9].

The search for wavelets with an "optimal" angular resolution lead to the Cauchy wavelet [6,10]. Its construction is based on minimizing an uncertainty relation, which measures the resolvable angular distance between different objects. The idea of using uncertanty principles for the construction of wavelets was investigated in [10], it uses the classical introduction of wavelet transforms via group representations[11,11,12,7,13,9] and their related Lie-algebras. The basic 1d-uncertainty principle which stems from the 1d-wavelet transform and the defining affine group is given by

$$\mu_\psi^2(A_b) \leq 4(\|\psi'\|^2 - \mu_\psi^2(A_b))(\|x\psi'\|_2^2 - \|\psi\|^2/4 - \mu_\psi^2(A_a))$$

the minimizing wavelet is $\psi = c(x-\lambda)^\alpha$.

This approach has been extended in several directions [14,9], a recent result in this field aims at combining the properties of windowed Fourier and wavelet transforms, the resulting transform exhibits a rather rich structure with several approaches for obtaining uncertainty principles and related wavelets,[15,16].

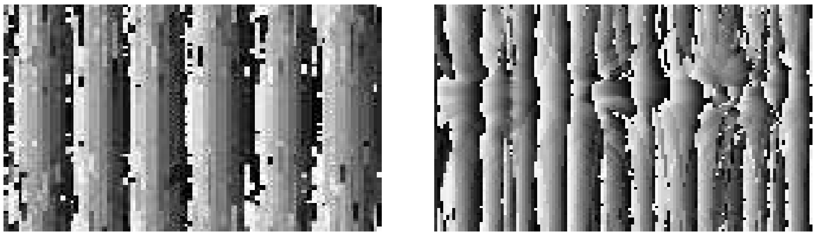

Figure 3: Phase representation of wavelet transformed Barkhausen data set.

3.2 phase information

The complex-valued Morlet wavelet is used whenever a Fourier-like multiscale decomposition of a signal f is required. The standard techniques for evaluating the resulting transformed data only uses the real-, imaginary or modulus of the transformed data. However, the phase information $\varphi(a,b)$ which is defined for 1d-signals by

$$L_\psi f(a,b) \;=\; e^{i\varphi(a,b)}\;|L_\psi f(a,b)|$$

exhibits significant structures, which are not fully understood yet. Prominent applications, which utilize phase information are the analysis of ECG-data, [17], and the classification of steel quality via Barkhausen measurements, [18], see Figure 3 , which exhibits the phase map of a Barkhausen data set.

The mathematical theory concerning the phase information of wavelet transforms goes back to Morlet, [19] and was further extended in [20]. These theoretical results are mainly concerned with the asymptotic behaviour of φ as a tends to zero and the investigation of the phase map for special function classes (progressive functions, Weierstrass function, etc).

3.3 Generalized indicator functions

Computing a Fourier or wavelet transform is only a first step for a quantitative analysis. The transformed data has to be further investigated and - in an optimal setting - to be reduced a single parameter which classifies the searched for quantity. The construction of such indicator functions for certain classes of signals and specific properties is basis for any quantitative wavelet method in signal and image processing.

The scale–energy spectrum of subsection 1.3 was a first example of an indicator function which determines dominant scales in a measured signal. A classical extension of this idea has been used to analyze fractal structures.

Fractal or multifractal structures are natural multiscale objects. They are therefore excellent candidates to demonstrate the usefulness of wavelet techniques.
The above defined indicator function can be refined by determining the decay rates for different energy measures:

$$n(a,q) \;=\; \int_{I\!R} |L_\psi f(a,b)|^q \; db \; .$$

The asymptotic behaviour as $a \to 0$

$$n(a,q) \;\sim\; a^{D(q)}$$

can be used detect and classify fractal and multifractal structures. For detailed examples and more information how to interpretate $D(q)$ see [21].

Recent approaches aim at incorporating the effect of the measurement process, i.e. they assume f itself is not available but rather a data set Af, where the operator A is the transfer function of the measurement device. The development and investigation of indicator functions, which are adapted to this situation, is one of the aims of the DFG-Schwerpunktprogramm "Mathematical methods for time series analysis and digital image processing", see http://www.math.uni-bremen.de/zetem/DFG-Schwerpunkt.

3.4 application I: analysis of earthquake data

The twodimensional continuous wavelet transform using wavelets with anisotropic shape and directional sensitivity (the Cauchy wavelets e. g., compare Antoine et al. [6]) as described in section 2.1 are an appropriate tool for directional analysis of twodimensional signals. Following the argumentation in section 1.2 wavelet extrema indicate a pattern matching for the associated transform parameters a, b, θ. And so in case of directional wavelets a match for rotation angle θ conforms with a directed feature of the signal along the direction θ. Further the location (according to b) and size (by the value of a) of that feature are determined.

If the wavelet is dilated by large scale values, then small adjacent features of the signal will be captured at once and cannot be distinguished by the wavelet. This makes use of the filter property of the wavelet transform as mentioned at section 1.5. That way chains of single features are collected to a large one. I. e. rotation and convolution could be used to detect onedimensional structures built by earthquakes and their aftershocks.

The spatial analysis of occurance of earthquakes and their aftershocks is based on earthquake catalogues. Each record of such data sets at least contains

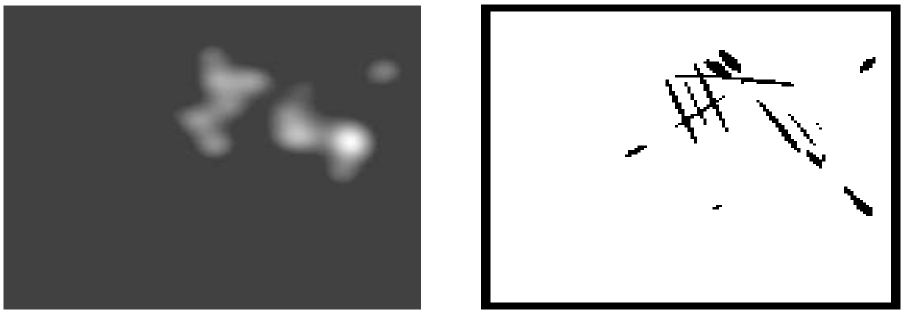

Figure 4: A gray-value representation of the seismic activity of the *Antofagasta catalogue* for the 18th August, 1995, 0:00 h using a model based on Gaussians to mark seismic events. Dark pixels represent small values and light pixels represent large values of the seismic activity (left) and marked areas of all directions detected by the Cauchy wavelet (imaginary part) at scale $a = 5.50969$ and date $d =$ `18th August, 1995, 0:00 h` (right)

the epicenter of the recorded activity in geographic coordinates, the date of appearance, its magnitude as well as the depth of the epicenter. Mapping each event to a spatial point makes it impossible to process such data by the integral transform (1). But earthquakes take effect in an area around of their epicenter. By a standard porcedure one constructs a continuous function to model the seismic activity, e.g. we use Gaussians as representatives for an event $\exp\{-[(x-x_i)^2-(y-y_i)^2]/2\}$, and sum them up in a certain time window to model the seismic activity of an earthquake area.

The aim of this investigation is to determine the critical subsurface structures, this preprocessing step allows to characterise the alignment of earthquakes and opens the possibility for a refined subsequent analysis of the spatio-temporal behaviour of earthquakes. Figure 4 shows the model function of the seismic activity constructed from the catalogue of the $\mathrm{M_w} = 8.0$ 1995 Antofagasta earthquake for a certain time window in a gray-value representation and the detected directional features. Latter was obtained by calculating the wavelet transform using the Cauchy wavelet and thresholding the wavelet coefficients relatively to the absolute maximum over all coefficients of each computed direction and location at a fixed scale. Then thresholded gray-value representations of each rotation angle were joined into a single image.

3.5 application II: wavelet filters for analyzing interferometric fringe patterns

Detection and classification of material faults is a major task in industrial quality control. In the field of non-destructive testing (NDT), several optical

a) b) c)

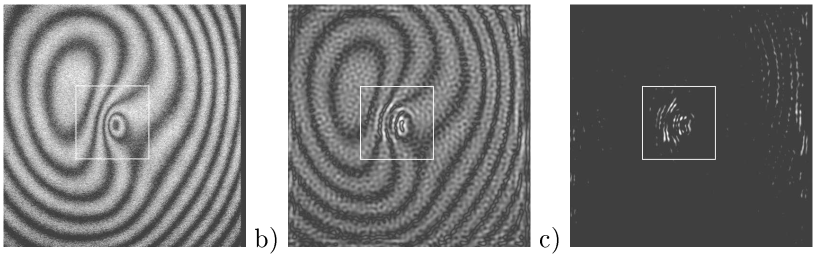

Figure 5: Application of an isotropic Morlet wavelet filter (b) and a post-processing transformation with an anisotropic gradient wavelet filter (c) to an "eye"-structure (a)

techniques – like shearography, classical and holographic interferometry – are applied and still are matter of investigation. For all these techniques interferometric fringe patterns (interferograms) are one of the results of the inspection process. Most of them are received by imposing a load (pressure, heat, etc.) to the object and by recording the dynamical change of the patterns during this process in real time. The data can be analyzed in a hybrid opto-electronic processor [22,23,24].

The different types of material flaw are indicated by typical sub-patterns which have been divided into five classes [25]. These sub-patterns can be extracted and distinguished from each other by the application of suiting continuous wavelet transformations [26]. The wavelet transformations are performed digitally or optically in the Fourier domain using the convolution theorem. One of those sub-patterns is the so called "eye"-structure (Fig. 5a). Subsequent application of an isotropic Morlet wavelet filter (Fig. 5b) and an anisotropic gradient wavelet filter (Fig. 5c) leads to the easy extraction of the "eye"-structure. The time-consuming part of this process is the trial of different scales in the continuous transformation.

During the dynamic process these sub-patterns change their exact appearance but not their typical shape. This behavior is exploited for the wavelet extraction process in the way that for the continuous transformation not the wavelet kernel is scaled, but the changing of the interferograms because of local variation of the fringe density is used for the matching (see Fig. 6). For the transformations the scale of the wavelet kernel is kept constant during the convolutions. The left part of every image in Fig. 6 shows the fringe pattern, the right one shows the transformation with an isotropic Morlet wavelet which is well suited for the extraction of the "eye"-structure in the upper image part.

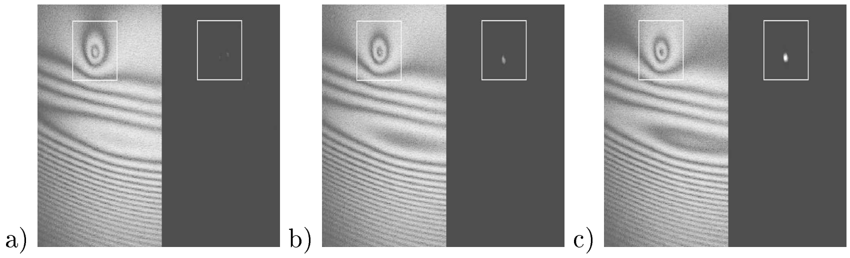

Figure 6: Wavelet transformation of a dynamically changing fringe pattern. The matching of the "eye"-structure in the data (left parts) changes from bad a) to very good c) in the transformations (right parts) which is indicated by the energy concentration in the region of the "eye".

While changing the load on the object, the obtained fringe pattern changes slightly. This results in no matching for Fig. 6a and good matching for Fig. 6c.

4 Conclusions

We have only sketched some wavelet techniques, for further and more refined methods the interested reader is referred to available textbooks, see e.g. [3,27,1,28]. Research articles on specific fields of applications are published e.g. in a wavelet series by Academic Press [29] or in the proceedings of the IEEE/SPIE conferences.

In particular we have not mentioned the application of wavelets for image compression, which is probably the most succesfull topic for wavelet analysis, see e.g. [30,31]. We have also neglected the applications of wavelet methods for solving operator equations and partial differential equations, see e.g. [32,33]. Finally we want to mention the "Wavelet Digest", the largest worldwide bulletin board for new wavelet developements, for subscritpion contact "wavelet@math.scarolina.edu".

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