THE CONTINUOUS SHEARLET TRANSFORM IN ARBITRARY SPACE DIMENSIONS

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ABSTRACT. This paper is concerned with the generalization of the continuous shearlet transform to higher dimensions. Similar to the two-dimensional case, our approach is based on translations, anisotropic dilations and specific shear matrices. We show that the associated integral transform again originates from a square-integrable representation of a specific group, the full n-variate shearlet group. Moreover, we verify that by applying the coorbit theory, canonical scales of smoothness spaces and associated Banach frames can be derived. We also indicate how our transform can be used to characterize singularities in signals.

1. INTRODUCTION

Modern technology allows for easy creation, transmission and storage of huge amounts of data. Confronted with a flood of data, such as internet traffic, or audio and video applications, nowadays the key problem is to extract the relevant information from these sets. To this end, usually the first step is to decompose the signal with respect to suitable building blocks which are well–suited for the specific application and allow a fast and efficient extraction. In this context, one particular problem which is currently in the center of interest is the analysis of *directional* information. Due to the bias to the coordinate axes, classical approaches such as, e.g., wavelet or Gabor transforms are clearly not the best choices, and hence new building blocks have to be developed. In recent studies, several approaches have been suggested such as ridgelets [2], curvelets [3], contourlets [8], shearlets [16] and many others. For a general approach see also [15]. Among all these approaches, the shearlet transform stands out because it is related to group theory, i.e., this transform can be derived from a square-integrable representation $\pi : \mathcal{S} \to \mathcal{U}(L_2(\mathbb{R}^2))$ of a certain group $\mathcal{S}$, the so-called *shearlet group*, see [6]. Therefore, in the context of the shearlet transform, all the powerful tools of group representation theory can be exploited.

So far, the shearlet transform is well developed for problems in $\mathbb{R}^2$. Given a shearlet $\psi \in L_2(\mathbb{R}^2)$, a signal $f \in L_2(\mathbb{R}^2)$ can be analyzed by its voice transform, the *shearlet transform*

$$\mathcal{SH}_\psi f(a,s,t) = \langle f, \pi(a,s,t)\,\psi(\cdot)\rangle = \langle f, |a|^{-\frac{3}{4}}\,\psi(A_a^{-1}S_s^{-1}(\cdot - t))\rangle, \tag{1}$$

where

$$A_a := \begin{pmatrix} a & 0 \\ 0 & \operatorname{sgn}(a)\sqrt{|a|} \end{pmatrix} \quad \text{and} \quad S_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \tag{2}$$

denote the *parabolic scaling matrix* and the *shear matrix*. By now, the continuous shearlet transform of two-dimensional functions is already well-established, however, for analyzing *higher-dimensional*

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data sets, there is clearly an urgent need for further generalizations. This is exactly the concern of this paper.

To our best knowledge, it seems that there exist only few results in this direction: some important progress has been achieved for the curvelet case in [1] and for surfacelets in [18]. However, for the shearlet approach the question is completely open.

The first step towards a higher-dimensional shearlet transform is the identification of a suitable shear matrix. Given a vector space V and a subspace $W \subset V$, a reasonable model reads as follows: the shear should fix the space W and translate all vectors parallel to W. That is, for $V = W \oplus W'$ and $v = w + w'$, the shear operation S can be described as $S(v) = w + (w' + M(w'))$ where M is a linear mapping from W' to W. Then S corresponds to a block matrix of the form

$$S = \begin{pmatrix} I & M \\ 0 & I \end{pmatrix}. \tag{3}$$

Then we are faced with the problem how to choose the block M. Since we want to end up with a square integrable group representation, one has to be careful. Usually, the number of parameters has to fit together with the space dimension, for otherwise the resulting group would be either to large or to small. Since we have n degrees of freedom related with the translates and one degree of freedom related with the dilation, $n-1$ degrees of freedom for the shear component would be optimal. Therefore one natural choice would be

$$S_s := \begin{pmatrix} 1 & s^{\mathrm{T}} \\ 0_{n-1} & I_{n-1} \end{pmatrix}. \tag{4}$$

Indeed, in Section 2 we show that with this choice the associated multivariate shearlet transform can be interpreted as a square integrable group representation of a $(2n)$-parameter group, the full shearlet group. It is a remarkable fact that the choice in (4) is in some sense a canonical one, other $(n-1)$-parameter choices might lead to nice group structures, but the representation will usually not be square integrable, see Remark 2.4 for details.

Once we have established a square integrable group representation, there is a very natural link to another useful concept, namely the coorbit space theory introduced by Feichtinger and Gröchenig in a series of papers [9, 10, 11, 12, 13]. By means of the coorbit space theory, it is possible to derive in a very natural way scales of smoothness spaces associated with the group representation. In this setting, the smoothness of functions is measured by the decay of the associated voice transform. Moreover, by a tricky discretization of the representation, it is possible to obtain (Banach) frames for these smoothness spaces. Fortunately, it turns out that for our multivariate continuous shearlet transform, all the necessary conditions for the application of the coorbit space theory can be established, so that we end up with new canonical smoothness spaces, the multivariate shearlet coorbit spaces, together with their discretizations.

One of the most important advantages of the two-dimensional continuous shearlet transform is the fact that it can be used to analyze singularities. Indeed, as outlined in [17], see also [4] for curvelets, it turns out that the decay of the continuous shearlet transform exactly describes the location and orientation of the singularities. By our approach these characterizations carry over to higher-dimensions.

This paper is organized as follows. in Section 2, we introduce the multivariate continuous shearlet transform and investigate its properties. We establish the full shearlet group and show that its representation in $L_2(\mathbb{R}^n)$ is indeed square integrable. Then, in Section 3, we discuss the relations with coorbit space theory. We show that all the necessary building block for the application of this theory can be provided. Consequently, the shearlet coorbit spaces and the associated families of Banach frames can be established. Finally, in Section 4, we investigate how the n-variate shearlet transform can be used to detect singularities in signals. It turn out that, similar to the two–dimensional case, the shape of the singularity is reflected by the decay of the shearlet transform.

2. Multivariate Continuous Shearlet Transform

In this section, we introduce the shearlet transform on $L_2(\mathbb{R}^n)$. This requires the generalization of the parabolic dilation matrix and of the shear matrix. Let I_n denote the (n,n)-identity matrix and 0_n, resp. 1_n the vectors with n entries 0, resp. 1. For $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}^{n-1}$, we set

$$A_a := \begin{pmatrix} a & 0_{n-1}^{\mathrm{T}} \\ 0_{n-1} & \operatorname{sgn}(a)|a|^{\frac{1}{n}} I_{n-1} \end{pmatrix} \quad \text{and} \quad S_s := \begin{pmatrix} 1 & s^{\mathrm{T}} \\ 0_{n-1} & I_{n-1} \end{pmatrix}.$$

Starting with shear transforms of the form (3), we will see in Remark 2.4 that our choice of S is somehow canonical if we want to relate our shearlet transform to a square integrable group representation. In order to have directional selectivity, the dilation factors at the diagonal of A_a should be chosen in an anisotropic way, i.e., if the first diagonal entry is a the other ones should increase less than linearly in a as $a \to \infty$. We will see that for our choice of A_a the shearlet transform for appropriate s and t increases at singularities as $a \to 0$, see Remark 4.2. We will use the relations

$$S_s^{-1} = \begin{pmatrix} 1 & -s^{\mathrm{T}} \\ 0_{n-1} & I_{n-1} \end{pmatrix} \quad \text{and} \quad S_s A_a S_{s'} A_{a'} = S_{s+|a|^{1-1/n}s'} A_{aa'}. \tag{5}$$

Lemma 2.1. *The set* $\mathbb{R}^* \times \mathbb{R}^{n-1} \times \mathbb{R}^n$ *endowed with the operation*

$$(a,s,t) \circ (a',s',t') = (aa', s + |a|^{1-1/n}\, s', t + S_s A_a t')$$

is a locally compact group $\mathbb{S}$ *which we call* full shearlet group. *The left and right Haar measures on* $\mathbb{S}$ *are given by*

$$d\mu_l(a,s,t) = \frac{1}{|a|^{n+1}}\, da\, ds\, dt \quad \text{and} \quad d\mu_r(a,s,t) = \frac{1}{|a|}\, da\, ds\, dt.$$

Proof. By the left relation in (5) it follows that $e := (1, 0_{n-1}, 0_n)$ is the neutral element in $\mathbb{S}$ and that the inverse of $(a,s,t) \in \mathbb{R}^* \times \mathbb{R}^{n-1} \times \mathbb{R}^n$ is given by

$$(a,s,t)^{-1} = (a^{-1}, -|a|^{1/n-1}\, s, -A_a^{-1} S_s^{-1} t).$$

The multiplication is associative since

$$\begin{aligned} \big((a,s,t) \circ (a',s',t')\big) \circ (a'',s'',t'') &= (aa', s + |a|^{1-1/n}s', t + S_s A_a t') \circ (a'',s'',t'') \\ = \quad & (aa'a'', s + |a|^{1-1/n}s' + |aa'|^{1-1/n}s'', t + S_s A_a t' + S_{s+|a|^{1-1/n}s'} A_{aa'} t'') \end{aligned}$$

and

$$\begin{aligned} (a,s,t) \circ \Big((a',s',t') \circ (a'',s'',t'')\Big) &= (a,s,t) \circ (a'a'', s' + |a'|^{1-1/n}s'', t' + S_{s'} A_{a'} t'') \\ = \quad & (aa'a'', s + |a|^{1-1/n}s' + |aa'|^{1-1/n}s'', t + S_s A_a t' + S_s A_a S_{s'} A_{a'} t'') \end{aligned}$$

coincide by the right equality in (5).

Further, we have for a function F on $\mathbb{S}$ that

$$\int_{\mathbb{S}} F\left((a',s',t') \circ (a,s,t)\right)\, d\mu_l(a,s,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} F(a'a, s' + |a'|^{1-1/n}s, t' + S_{s'} A_{a'} t)\, dt\, ds\, \frac{da}{|a|^{n+1}}$$

and by substituting $\tilde{t} := t' + S_{s'} A_{a'} t$, i.e., $d\tilde{t} = |a'|^{2-1/n}\, dt$ $\tilde{s} := s' + |a'|^{1-1/n} s$, i.e., $d\tilde{s} = |a'|^{(n-1)^2/n}\, ds$ and $\tilde{a} := a'a$ that

$$\int_{\mathbb{S}} F\left((a',s',t') \circ (a,s,t)\right)\, d\mu_l(a,s,t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} F(\tilde{a},\tilde{s},\tilde{t})\, \frac{1}{|a'|^{2-1/n}} \frac{|a'|^{n+1}}{|a'|^{(n-1)^2/n}} \frac{1}{|a'|} \frac{d\tilde{a}}{|\tilde{a}|^{n+1}} d\tilde{s}\, d\tilde{t}$$

so that $d\mu_l$ is indeed the left Haar measure on $\mathbb{S}$. Similarly we can verify that $d\mu_r$ is the right Haar measure on $\mathbb{S}$. □

In the following, we use only the left Haar measure and use the abbreviation $d\mu = d\mu_l$. For $f \in L_2(\mathbb{R}^n)$ we define

$$\pi(a,s,t) f(x) = f_{a,s,t}(x) := |a|^{\frac{1}{2n}-1} f(A_a^{-1} S_s^{-1}(x-t)). \tag{6}$$

It is easy to check that $\pi : \mathbb{S} \to \mathcal{U}(L_2(\mathbb{R}^n))$ is a mapping from $\mathbb{S}$ into the group $\mathcal{U}(L_2(\mathbb{R}^n))$ of unitary operators on $L_2(\mathbb{R}^n)$. The *Fourier transform* of $f_{a,s,t}$ is given by

$$\begin{aligned} \hat{f}_{a,s,t}(\omega) &= \int_{\mathbb{R}^n} f_{a,s,t}(x) e^{-2\pi i \langle x,\omega\rangle}\, dx \\ &= |a|^{1-\frac{1}{2n}}\, e^{-2\pi i \langle t,\omega\rangle}\, \hat{f}(A_a^{\mathrm{T}} S_s^{\mathrm{T}} \omega) \\ &= |a|^{1-\frac{1}{2n}}\, e^{-2\pi i \langle t,\omega\rangle}\, \hat{f}\begin{pmatrix} a\omega_1 \\ \operatorname{sgn}(a)|a|^{\frac{1}{n}}(\omega_1 s + \tilde{\omega}) \end{pmatrix} \end{aligned} \tag{7}$$

where $\omega = (\omega_1, \tilde{\omega}^{\mathrm{T}})^{\mathrm{T}}$.

Recall that a *unitary representation* of a locally compact group G with the left Haar measure μ on a Hilbert space $\mathcal{H}$ is a homomorphism π from G into the group of unitary operators $\mathcal{U}(\mathcal{H})$ on $\mathcal{H}$ which is continuous with respect to the strong operator topology.

Lemma 2.2. *The mapping π defined by* (6) *is a unitary representation of* $\mathbb{S}$.

Proof. Let $\psi \in L^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and $(a,s,t), (a',s',t') \in \mathbb{S}$. Using (5) we obtain

$$\begin{aligned} \pi(a,s,t)(\pi(a',s',t')\psi)(x) &= |a|^{\frac{1}{2n}-1} \pi(a',s',t')\psi(A_a^{-1} S_s^{-1}(x-t)) \\ &= |aa'|^{\frac{1}{2n}-1} \psi(A_{a'}^{-1} S_{s'}^{-1}(A_a^{-1} S_s^{-1}(x-t) - t')) \\ &= |aa'|^{\frac{1}{2n}-1} \psi(A_{a'}^{-1} S_{s'}^{-1} A_a^{-1} S_s^{-1}(x - (t + S_s A_a t'))) \\ &= |aa'|^{\frac{1}{2n}-1} \psi(A_{aa'}^{-1} S_{s+|a|^{1-\frac{1}{n}} s'}^{-1}(x - (t + S_s A_a t'))) \\ &= \pi((a,s,t) \circ (a',s',t'))\psi(x). \end{aligned}$$

□

A nontrivial function $\psi \in L_2(\mathbb{R}^n)$ is called *admissible*, if

$$\int_{\mathbb{S}} |\langle \psi, \pi(a,s,t)\psi \rangle|^2 d\mu(a,s,t) < \infty.$$

If π is irreducible and there exits at least one admissible function $\psi \in L_2(\mathbb{R}^n)$, then π is called *square integrable.* The following result shows that the unitary representation π defined in (6) is square integrable.

Theorem 2.3. *A function $\psi \in L_2(\mathbb{R}^n)$ is admissible if and only if it fulfills the* admissibility condition

$$C_\psi := \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\omega)|^2}{|\omega_1|^n}\, d\omega < \infty. \tag{8}$$

Then, for any $f \in L^2(\mathbb{R}^n)$, the following equality holds true:

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2 \, d\mu(a,s,t) = C_\psi \, \|f\|^2_{L_2(\mathbb{R}^n)} \,. \tag{9}$$

In particular, the unitary representation π is irreducible and hence square integrable.

Proof. Employing the Plancherel theorem and (7), we obtain

$$\begin{aligned}
\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2 \, d\mu(a,s,t) &= \int_{\mathbb{S}} |f * \psi^*_{a,s,0}(t)|^2 \, dt \, ds \, \frac{da}{|a|^{n+1}} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |\hat{f}(\omega)|^2 |\hat{\psi^*}_{a,s,0}(\omega)|^2 \, d\omega \, ds \, \frac{da}{|a|^{n+1}} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |\hat{f}(\omega)|^2 |a|^{2-\frac{1}{n}} |\hat{\psi}(A_a^{\mathrm{T}} S_s^{\mathrm{T}} \omega)|^2 \, d\omega \, ds \, \frac{da}{|a|^{n+1}} \qquad (10) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} |\hat{f}(\omega)|^2 |a|^{1-n-\frac{1}{n}} |\hat{\psi}\begin{pmatrix} a\omega_1 \\ \operatorname{sgn}(a)|a|^{\frac{1}{n}}(\tilde{\omega} + \omega_1 s)\end{pmatrix}|^2 \, ds \, d\omega \, da,
\end{aligned}$$

where $\psi^*_{a,s,0}(x) = \overline{\psi_{a,s,0}(-x)}$. Substituting $\tilde{\xi} := \operatorname{sgn}(a)|a|^{\frac{1}{n}}(\tilde{\omega} + \omega_1 s)$, i.e., $(|a|^{\frac{1}{n}}\omega_1)^{n-1} \, ds = d\tilde{\xi}$, we obtain

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2 \; d\mu(a,s,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} |\hat{f}(\omega)|^2 \, |a|^{-n} \, |\omega_1|^{-(n-1)} \, |\hat{\psi}\begin{pmatrix} a\omega_1 \\ \tilde{\xi}\end{pmatrix}|^2 \, d\tilde{\xi} \, d\omega \, da.$$

Next, we substitute $\xi_1 := a\omega_1$, i.e., $\omega_1 \, da = d\xi_1$ which results in

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2 \; d\mu(a,s,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} |\hat{f}(\omega)|^2 \, \frac{|\omega_1|^n}{|\xi_1|^n |\omega_1|^n} \, |\hat{\psi}\begin{pmatrix} \xi_1 \\ \tilde{\xi}\end{pmatrix}|^2 \, d\tilde{\xi} \, d\omega \, d\xi_1 = C_\psi \, \|f\|^2_{L_2(\mathbb{R}^n)}.$$

Setting $f := \psi$, we see that ψ is admissible if and only if C_ψ is finite.

The irreducibility of π follows from (9) in the same way as in [5]. ☐

By the following remark, the choice of the shear matrix S_s is canonical to ensure that π is a square integrable representation.

Remark 2.4. Assume that our shear matrix has the form (3) with $M = (m_{ij})_{i,j=1}^{p,n-p} \in \mathbb{R}^{p,n-p}$. Let M contain N different entries (variables). We assume that $N \geq n-1$ since we have one dilation parameter and otherwise the group becomes too small. Then we obtain instead of (10)

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2 \, d\mu(a,s,t) \;\; = \;\; \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} |\hat{f}(\omega)|^2 |a|^{1-n-\frac{1}{n}} |\hat{\psi}(A_a \begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 + M^{\mathrm{T}}\tilde{\omega}_1 \end{pmatrix})|^2 \, ds \, d\omega \, da, \tag{11}$$

where $\tilde{\omega}_1 := (\omega_1, \ldots, \omega_p)^{\mathrm{T}}$ and $\tilde{\omega}_2 := (\omega_{p+1}, \ldots, \omega_n)^{\mathrm{T}}$. Now we can use the following substitution procedure:

$$\xi_{p+1} := |a|^{1/n}(\omega_{p+1} + m_{11}\omega_1 + \ldots + m_{p1}\omega_p), \tag{12}$$

i.e., $d\xi_{p+1} = |\omega_1| dm_{11}$ and with corresponding modifications if some of the m_{j1}, $j > 1$ are the same as m_{11}. Then we replace m_{11} in the other rows of $\tilde{\omega}_2 + M^{\mathrm{T}}\tilde{\omega}_1$ where it appears by (12). Next we continue to substitute the second row if it contains an integration variable from M ($\neq m_{11}$). Continuing this substitution process up to the final row we have at the end replaced the lower $n-p$ values in $\hat{\psi}$ by $n-q$, $q \leq p$ variables $\xi_1 = \xi_{j_1}, \ldots, \xi_{j_{n-q}}$ and some functions depending only on $a, \omega, \xi_{j_1}, \ldots, \xi_{j_{n-q}}$. Consequently, the integrand depends only on these variables. However, we have to integrate over $a, \omega, \xi_{j_1}, \ldots, \xi_{j_{n-q}}$ and over the remaining $N - (n-q)$ variables from M. But then the integral in (11) becomes infinity unless $N = n-q$. Since $n-1 \leq N$ this implies $q = p = 1$, i.e., our choice of S_s.

A function $\psi \in L_2(\mathbb{R}^n)$ fulfilling the admissibility condition (8) is called a *continuous shearlet* and the transform $\mathcal{SH}_\psi : L_2(\mathbb{R}^n) \to L_2(\mathbb{S})$,

$$\mathcal{SH}_\psi f(a,s,t) := \langle f, \psi_{a,s,t}\rangle = (f * \psi^*_{a,s,0})(t), \tag{13}$$

continuous shearlet transform.

Remark 2.5. An example of a continuous shearlet can be constructed as follows: Let ψ_1 be a continuous wavelet with $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ and supp $\hat{\psi}_1 \subseteq [-2,-\frac{1}{2}] \cup [\frac{1}{2},2]$, and let ψ_2 be such that $\hat{\psi}_2 \in C^\infty(\mathbb{R}^{n-1})$ and supp $\hat{\psi}_2 \subseteq [-1,1]^{n-1}$. Then the function $\psi \in L^2(\mathbb{R}^n)$ defined by

$$\hat{\psi}(\omega) = \hat{\psi}(\omega_1, \tilde{\omega}) = \hat{\psi}_1(\omega_1)\, \hat{\psi}_2\left(\frac{1}{\omega_1}\,\tilde{\omega}\right)$$

is a continuous shearlet. The support of $\hat{\psi}$ is depiced for $\omega_1 \geq 0$ in Fig. 1.

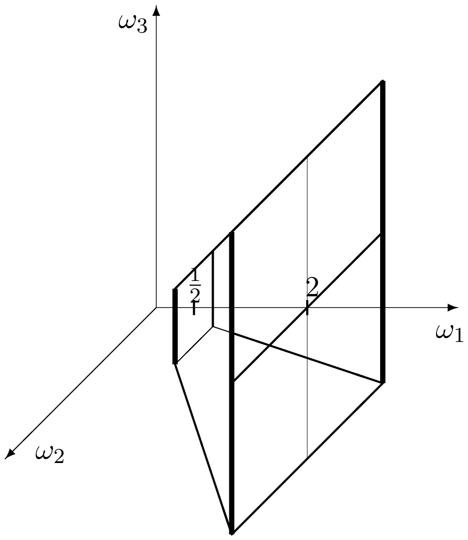

FIGURE 1. Support of the shearlet $\hat{\psi}$ in Remark 2.5 for $\omega_1 \geq 0$.

3. MULTIVARIATE SHEARLET COORBIT THEORY

In this section we want to establish a coorbit theory based on the square integrable representation (6) of the shearlet group. We mainly follow the lines of [5]. For further information on coorbit space theory, the reader is referred to [9, 10, 11, 12, 13].

3.1. **Shearlet Coorbit Space.** We consider weight functions $w(a,s,t) = w(a,s)$ that are locally integrable with respect to a and s, i.e., $w \in L_1^{loc}(\mathbb{R}^n)$ and fulfill $w\left((a,s,t)\circ(a',s',t')\right) \leq w(a,s,t)w(a',s',t')$ and $w(a,s,t) \geq 1$ for all $(a,s,t),(a',s',t') \in \mathbb{S}$. For $1 \leq p < \infty$, let

$$L_{p,w}(\mathbb{S}) := \{F \text{ measurable on } \mathbb{S} : \|F\|_{L_{p,w}(\mathbb{S})} := \left(\int_{\mathbb{S}} |F(g)|^p\, w(a,s,t)^p d\mu(a,s,t)\right)^{1/p} < \infty\},$$

and let $L_{\infty,w}$ be defined with the usual modifications. In order to construct the coorbit spaces related to the shearlet group we have to ensure that there exists a function $\psi \in L_2(\mathbb{R}^n)$ such that

$$\mathcal{SH}_\psi(\psi) = \langle \psi, \pi(a,s,t)\psi\rangle \in L_{1,w}(\mathbb{S}). \tag{14}$$

To this end, we need a preliminary lemma on the support of ψ.

Lemma 3.1. *Let $a_1 > a_0 \geq \alpha > 0$ and $b = (b_1, \dots, b_{n-1})^T$ be a vector with positive components. Suppose that* $\operatorname{supp} \hat{\psi} \subseteq ([-a_1, -a_0] \cup [a_0, a_1]) \times Q_b$, *where $Q_b := [-b_1, b_1] \times \cdots \times [-b_{n-1}, b_{n-1}]$. Then $\hat{\psi}\hat{\psi}_{a,s,0} \not\equiv 0$ implies $a \in \big[-\frac{a_1}{a_0}, -\frac{a_0}{a_1}\big] \cup \big[\frac{a_0}{a_1}, \frac{a_1}{a_0}\big]$ and $s \in Q_c$, where $c := \frac{1+(a_1/a_0)^{1/n}}{a_0}\, b$.*

Proof. Let us first dicusss the case $a > 0$. By (7) we see that the following conditions are necessary for $\hat{\psi}(\omega)\hat{\psi}_{a,s,0}(\omega) \not\equiv 0$:

$$\begin{array}{cccccc} & a_0 \leq \omega_1 \leq a_1 & \text{and} & \frac{a_0}{a} \leq \omega_1 \leq \frac{a_1}{a} & & \text{or} \\ \text{i)} & & & & & \\ & -a_1 \leq \omega_1 \leq -a_0 & \text{and} & -\frac{a_1}{a} \leq \omega_1 \leq -\frac{a_0}{a}, & & \\ \text{ii)} & -b \leq \tilde{\omega} \leq b & \text{and} & -a^{-1/n} b - \omega_1 s \leq \tilde{\omega} \leq a^{-1/n} b - \omega_1 s & & \end{array}$$

where $\tilde{\omega} := (\omega_2, \dots, \omega_{n-1})^{\mathrm{T}}$ and ii) is meant componentwise. Condition i) implies that

$$a \in \Big[\frac{a_0}{a_1}, \frac{a_1}{a_0}\Big]. \tag{15}$$

For $s_i \geq 0$ and $a_0 \leq \omega_1 \leq a_1$ the second condition in ii) becomes

$$-a^{-1/n} b_i - s_i a_1 \leq \omega_{i+1} \leq a^{-1/n} b_i - s_i a_0$$

and with (15) further

$$-\left(\frac{a_0}{a_1}\right)^{-1/n} b_i - s_i a_1 \leq \omega_{i+1} \leq \left(\frac{a_0}{a_1}\right)^{-1/n} b_i - s_i a_0.$$

Together with the first condition in ii) this results in $s_i \leq \frac{b_i}{a_0}\big(1 + \big(\frac{a_1}{a_0}\big)^{1/n}\big)$. The same condition can be deduced for $s_i \geq 0$ and $-a_1 \leq \omega_1 \leq -a_0$.

For $s_i < 0$ and $a_0 \leq \omega_1 \leq a_1$ or $-a_1 \leq \omega_1 \leq -a_0$, we obtain that $s_i \geq -\frac{b_i}{a_0}\big(1 + \big(\frac{a_1}{a_0}\big)^{1/n}\big)$ is necessary for $\hat{\psi}(\omega)\hat{\psi}_{a,s,0}(\omega) \not\equiv 0$.

Finally, the case $a < 0$ can be treated similarly which results in $a \in \big[-\frac{a_1}{a_0}, -\frac{a_0}{a_1}\big]$. This completes the proof. ☐

Now we can prove the required property (14) of $\mathcal{SH}_\psi(\psi)$.

Theorem 3.2. *Let ψ be a Schwartz function such that* $\operatorname{supp} \hat{\psi} \subseteq ([-a_1, -a_0] \cup [a_0, a_1]) \times Q_b$. *Then we have that $\mathcal{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$, i.e.,*

$$\|\langle \psi, \pi(\cdot)\psi \rangle\|_{L_{1,w}(\mathbb{S})} = \int_{\mathbb{S}} |\mathcal{SH}_\psi(\psi)(a,s,t)|\, w(a,s,t)\, d\mu(a,s,t) < \infty.$$

Proof. Straightforward computation gives

$$\begin{aligned} \|\langle \psi, \pi(\cdot)\psi \rangle\|_{L_{1,w}(\mathbb{S})} &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |\langle \psi, \psi_{a,s,t} \rangle|\, w(a,s)\, dt ds \frac{da}{|a|^{n+1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |\psi * \psi^*_{a,s,0}(t)|\, w(a,s)\, dt ds \frac{da}{|a|^{n+1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |\mathcal{F}^{-1}\mathcal{F}\big(\psi * \psi^*_{a,s,0}\big)(t)|\, dt\, w(a,s)\, ds \frac{da}{|a|^{n+1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \|\mathcal{F}\big(\psi * \psi^*_{a,s,0}\big)\|_{\mathcal{F}^{-1}L_1}\, w(a,s)\, ds \frac{da}{|a|^{n+1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \|\hat{\psi}\, \bar{\hat{\psi}}_{a,s,0}\|_{\mathcal{F}^{-1}L_1}\, w(a,s)\, ds \frac{da}{|a|^{n+1}}, \end{aligned}$$

where $\|f\|_{\mathcal{F}^{-1}L_1(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\mathcal{F}^{-1}f(x)|\,dx$ for $f \in L_1(\mathbb{R}^n)$. By Lemma 3.1 this can be rewritten as

$$\|\langle\psi,\pi(\cdot)\psi\rangle\|_{L_{1,w}(\mathbb{S})} = \left(\int_{-a_1/a_0}^{-a_0/a_1} + \int_{a_0/a_1}^{a_1/a_0}\right)\int_{Q_c} \|\hat{\psi}\,\hat{\psi}^*_{a,s,0}\|_{\mathcal{F}^{-1}L_1(\mathbb{R}^n)}\, w(a,s)\,ds\,\frac{da}{|a|^{n+1}},$$

which is obviously finite. ☐

For ψ satisfying (14) we can consider the space

$$\mathcal{H}_{1,w} := \{f \in L_2(\mathbb{R}^n) : \mathcal{SH}_\psi(f) = \langle f,\pi(\cdot)\psi\rangle \in L_{1,w}(\mathbb{S})\}, \tag{16}$$

with norm $\|f\|_{\mathcal{H}_{1,w}} := \|\mathcal{SH}_\psi f\|_{L_{1,w}(\mathbb{S})}$ and its anti-dual $\mathcal{H}^\sim_{1,w}$, the space of all continuous conjugate-linear functionals on $\mathcal{H}_{1,w}$. The spaces $\mathcal{H}_{1,w}$ and $\mathcal{H}^\sim_{1,w}$ are π-invariant Banach spaces with continuous embeddings $\mathcal{H}_{1,w} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^\sim_{1,w}$, and their definition is independent of the shearlet ψ. Then the inner product on $L_2(\mathbb{R}^n) \times L_2(\mathbb{R}^n)$ extends to a sesquilinear form on $\mathcal{H}^\sim_{1,w} \times \mathcal{H}_{1,w}$, therefore for $\psi \in \mathcal{H}_{1,w}$ and $f \in \mathcal{H}^\sim_{1,w}$ the *extended representation coefficients*

$$\mathcal{SH}_\psi(f)(a,s,t) := \langle f,\pi(a,s,t)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}}$$

are well-defined. Now, for $1 \le p \le \infty$, we define the *shearlet coorbit spaces*

$$\mathcal{SC}_{p,w} := \{f \in \mathcal{H}^\sim_{1,w} : \mathcal{SH}_\psi(f) \in L_{p,w}(\mathbb{S})\} \tag{17}$$

with norms $\|f\|_{\mathcal{SC}_{p,w}} := \|\mathcal{SH}_\psi f\|_{L_{p,w}(\mathbb{S})}$. It holds that $\mathcal{SC}_{1,w} = \mathcal{H}_{1,w}$ and $\mathcal{SC}_{1,1} = L_2(\mathbb{R}^n)$.

3.2. **Shearlet Banach Frames.** The Feichtinger-Gröchenig theory provides us with a machinery to construct atomic decompositions and Banach frames for our shearlet coorbit spaces $\mathcal{SC}_{p,w}$. In a first step, we have to determine, for a compact neighborhood U of $e \in \mathbb{S}$ with non-void interior, so-called U–dense sets. A (countable) family $X = ((a,s,t)_\lambda)_{\lambda\in\Lambda}$ in $\mathbb{S}$ is said to be *U-dense* if $\cup_{\lambda\in\Lambda}(a,s,t)_\lambda U = \mathbb{S}$, and *separated* if for some compact neighborhood Q of e we have $(a_i,s_i,t_i)Q \cap (a_j,s_j,t_j)Q = \emptyset, i \neq j$, and *relatively separated* if X is a finite union of separated sets.

Lemma 3.3. *Let U be a neighborhood of the identity in $\mathbb{S}$, and let $\alpha > 1$ and $\beta,\gamma > 0$ be defined such that*

$$[\alpha^{\frac{1}{n}-1},\alpha^{\frac{1}{n}}) \times [-\tfrac{\beta}{2},\tfrac{\beta}{2})^{n-1} \times [-\tfrac{\gamma}{2},\tfrac{\gamma}{2})^n \subseteq U. \tag{18}$$

Then the sequence

$$\{(\epsilon\alpha^j, \beta\alpha^{j(1-\frac{1}{n})}k, S_{\beta\alpha^{j(1-\frac{1}{n})}k}A_{\alpha^j}\gamma m) : j \in \mathbb{Z}, k \in \mathbb{Z}^{n-1}, m \in \mathbb{Z}^n,\ \epsilon \in \{-1,1\}\} \tag{19}$$

is U-dense and relatively separated.

Proof. Set

$$U_0 := [\alpha^{\frac{1}{n}-1},\alpha^{\frac{1}{n}}) \times [-\tfrac{\beta}{2},\tfrac{\beta}{2})^{n-1} \times [-\tfrac{\gamma}{2},\tfrac{\gamma}{2})^n.$$

It is sufficient to prove that the sequence (19) is U_0-dense.

For this, fix any $(x,y,z) \in \mathbb{S}$. In the following we assume that $x \in \mathbb{R}^+$ in which case we have to set $\epsilon = 1$. If $x < 0$, the same arguments apply while choosing $\epsilon = -1$. We have that

$$\begin{aligned}(\alpha^j, \beta\alpha^{j(1-\frac{1}{n})}k, S_{\beta\alpha^{j(1-\frac{1}{n})}k}A_{\alpha^j}\gamma m) \circ U_0 &= \{(\alpha^j u, \alpha^{j(1-\frac{1}{n})}(\beta k + v), S_{\beta\alpha^{j(1-\frac{1}{n})}k}A_{\alpha^j}(\gamma m + w)) : \\ &\qquad (u,v,w) \in U_0\}.\end{aligned}$$

Then $[\log_\alpha x + (\frac{1}{n} - 1), \log_\alpha x + \frac{1}{n})$ contains a unique integer j, and there exists a unique $u \in [\alpha^{\frac{1}{n}-1}, \alpha^{\frac{1}{n}})$ such that $\log_\alpha x = \log_\alpha u + j$. Further, there exist unique $k \in \mathbb{Z}^{n-1}$ and $v \in [-\frac{\beta}{2}, \frac{\beta}{2})^{n-1}$ so that $\beta k + v = \alpha^{j(\frac{1}{n}-1)} y$. Finally, we have that

$$S_{\beta\alpha^{j(1-\frac{1}{n})}k} A_{\alpha^j}(\gamma m + w) = \begin{pmatrix} \alpha^j \left(\gamma m_1 + w_1\right) + \beta\langle k, \gamma\tilde{m} + \tilde{w}\rangle)) \\ \alpha^{\frac{j}{n}}(\gamma\tilde{m} + \tilde{w}) \end{pmatrix}.$$

There exist unique $\tilde{m} \in \mathbb{Z}^{n-1}$ and $\tilde{w} \in [-\frac{\gamma}{2}, \frac{\gamma}{2})^{n-1}$ such that $\gamma\tilde{m} + \tilde{w} = \alpha^{-\frac{j}{n}}\tilde{z}$ and unique $m_1 \in \mathbb{Z}$ and $w_1 \in [-\frac{\gamma}{2}, \frac{\gamma}{2})$ such that $\gamma m_1 + w_1 = \alpha^{-j} z_1 - \beta\alpha^{-\frac{j}{n}}\langle k, \tilde{z}\rangle$. We have shown that there exist unique $(u, v, w) \in U_0$ and $j \in \mathbb{Z}$, $k \in \mathbb{Z}^{n-1}$, $m \in \mathbb{Z}^n$ such that

$$(x, y, z) = (\alpha^j u, \alpha^{j(1-\frac{1}{n})}(\beta k + v), S_{\beta\alpha^{j(1-\frac{1}{n})}k} A_{\alpha^j}(\gamma m + w)).$$

Finally, the uniqueness of the decomposition proves immediately that the chosen sequence is relatively separated. □

Next we define the *U–oscillation* as

$$\mathrm{osc}_U(a, s, t) := \sup_{u \in U} |\mathcal{SH}_\psi(\psi)(u \circ (a, s, t)) - \mathcal{SH}_\psi(\psi)(a, s, t)|. \tag{20}$$

Then, the following decomposition theorem, which was proved in a general setting in [9, 10, 11, 12, 13], says that discretizing the representation by means of an U-dense set produces an atomic decomposition for $\mathcal{SC}_{p,w}$.

Theorem 3.4. *Assume that the irreducible, unitary representation π is w-integrable and let an appropriately normalized $\psi \in L_2(\mathbb{R}^n)$ which fulfills*

$$M\langle\psi, \pi(a, s, t)\rangle := \sup_{u \in (a,s,t)U} |\langle\psi, \pi(u)\psi\rangle| \in L_{1,w}(\mathbb{S}) \tag{21}$$

be given. Choose a neighborhood U of e so small that

$$\|\,\mathrm{osc}_U\,\|_{L_{1,w}(\mathbb{S})} < 1. \tag{22}$$

Then for any U-dense and relatively separated set $X = ((a, s, t)_\lambda)_{\lambda\in\Lambda}$ the space $\mathcal{SC}_{p,w}$ has the following atomic decomposition*: If $f \in \mathcal{SC}_{p,w}$, then*

$$f = \sum_{\lambda\in\Lambda} c_\lambda(f)\pi((a, s, t)_\lambda)\psi \tag{23}$$

where the sequence of coefficients depends linearly on f and satisfies

$$\|(c_\lambda(f))_{\lambda\in\Lambda}\|_{\ell_{p,w}} \le C\|f\|_{\mathcal{SC}_{p,w}} \tag{24}$$

with a constant C depending only on ψ and with $\ell_{p,w}$ being defined by

$$\ell_{p,w} := \{c = (c_\lambda)_{\lambda\in\Lambda} : \|c\|_{\ell_{p,w}} := \|cw\|_{\ell_p} < \infty\},$$

where $w = (w((a, s, t)_\lambda))_{\lambda\in\Lambda}$. Conversely, if $(c_\lambda(f))_{\lambda\in\Lambda} \in \ell_{p,w}$, then $f = \sum_{\lambda\in\Lambda} c_\lambda\pi((a, s, t)_\lambda)\psi$ is in $\mathcal{SC}_{p,w}$ and

$$\|f\|_{\mathcal{SC}_{p,w}} \le C'\|(c_\lambda(f))_{\lambda\in\Lambda}\|_{\ell_{p,w}}. \tag{25}$$

Given such an atomic decomposition, the problem arises under which conditions a function f is completely determined by its *moments* $\langle f, \pi((a, s, t)_\lambda)\psi\rangle$ and how f can be reconstructed from these moments. This is answered by the following theorem which establishes the existence of Banach frames.

Theorem 3.5. *Impose the same assumptions as in Theorem 3.4. Choose a neighborhood U of e such that*

$$\|\operatorname{osc}_U\|_{L_{1,w}(\mathbb{S})} < 1/\|\mathcal{SH}_\psi(\psi)\|_{L_{1,w}(\mathbb{S})}. \tag{26}$$

Then, for every U-dense and relatively separated family $X = ((a,s,t)_\lambda)_{\lambda\in\Lambda}$ in G the set $\{\pi((a,s,t)_\lambda)\psi : \lambda \in \Lambda\}$ is a Banach frame for $\mathcal{SH}_{p,w}$. This means that

- i) *$f \in \mathcal{SC}_{p,w}$ if and only if $(\langle f, \pi((a,s,t)_\lambda)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda} \in \ell_{p,w}$;*
- ii) *there exist two constants $0 < D \le D' < \infty$ such that*

$$D\,\|f\|_{\mathcal{SC}_{p,w}} \le \|(\langle f, \pi((a,s,t)_\lambda)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda}\|_{\ell_{p,w}} \le D'\,\|f\|_{\mathcal{SC}_{p,w}}; \tag{27}$$

- iii) *there exists a bounded, linear reconstruction operator $\mathcal{S}$ from $\ell_{p,w}$ to $\mathcal{SC}_{p,w}$ such that $\mathcal{S}\left((\langle f, \psi((a,s,t)_\lambda)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda}\right) = f$.*

It remains to check how the conditions (21), (22) and (26) can be ensured. To this end, we need the following lemma which was proved in a general setting in [13].

Lemma 3.6. *Let $\mathcal{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$ and $\operatorname{osc}_U \in L_{1,w}(\mathbb{S})$ for one compact neighborhood U of e. Then we have that ψ fulfills (21). If, in addition, $\mathcal{SH}_\psi(\psi)$ is continuous, then*

$$\lim_{U\to\{e\}} \|\operatorname{osc}_U\|_{L_{1,w}(\mathbb{S})} = 0. \tag{28}$$

To apply the whole machinery of Theorems 3.4 and 3.5 to our shearlet group setting it remains to prove that $\|\operatorname{osc}_U\|_{L_{1,w}(\mathbb{S})}$ becomes arbitrarily small for a sufficiently small neighborhood U of e.

Theorem 3.7. *Let ψ be a function contained in the Schwartz space $\mathcal{S}$ with* $\operatorname{supp}\hat\psi \subseteq ([-a_1,-a_0] \cup [a_0,a_1]) \times Q_b$. *Then, for every $\varepsilon > 0$, there exists a sufficiently small neighborhood U of e so that*

$$\|\operatorname{osc}_U\|_{L_{1,w}(\mathbb{S})} \le \varepsilon. \tag{29}$$

Proof. By Theorem 3.2 we have that $\mathcal{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$. Moreover, it is easy to check that $\mathcal{SH}_\psi(\psi)$ is continuous on $\mathbb{S}$. Thus, by Lemma 3.6, it remains to show that $\operatorname{osc}_U \in L_{1,w}(\mathbb{S})$ for some compact neighborhood of e. By definition of osc_U and Parseval's identity we have that

$$\begin{aligned}\operatorname{osc}_U(a,s,t) &= \sup_{(\alpha,\beta,\gamma)\in U} \left|\langle\hat\psi, \hat\psi_{a,s,t}\rangle - \langle\hat\psi, \hat\psi_{(\alpha,\beta,\gamma)(a,s,t)}\rangle\right| \\ &= \sup_{(\alpha,\beta,\gamma)\in U} \left||a|^{1-\frac{1}{2n}}\mathcal{F}\left(\hat\psi(A_a S_s^{\mathrm{T}}\,\cdot)\bar{\hat\psi}\right)(t) - |a\alpha|^{1-\frac{1}{2n}}\mathcal{F}\left(\hat\psi(A_{a\alpha}S^{\mathrm{T}}_{\beta+\alpha^{1-1/n}s}\,\cdot)\bar{\hat\psi}\right)(\gamma + S_\beta A_\alpha t)\right|,\end{aligned}$$

where we can assume that $\alpha > 0$. By Lemma 3.1, we see that for (α,β) in a sufficiently small neighborhood of $(1, 0_{n-1})$, the function $\hat\psi(A_{a\alpha}S^{\mathrm{T}}_{\beta+\alpha^{1-1/n}s}\,\cdot)\bar{\hat\psi}$ becomes zero except for values a contained in two finite intervals away from zero and values s in a finite interval. Thus, it remains to show that $\int_{\mathbb{R}^n} \operatorname{osc}_U(a,s,t)\,dt \le C(a,s)$ with a finite constant $C(a,s)$. The main idea is to split the integral into three parts

$$\int_{\mathbb{R}^n} \operatorname{osc}_U(a,s,t)\,dt = |a|^{1-\frac{1}{2n}}(I_1 + I_2 + I_3),$$

where

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}^n} \sup_{(\alpha,\beta,\gamma)\in U} \big||1-\alpha^{1-\frac{1}{2n}}|\mathcal{F}\left(\hat{\psi}(A_a S_s^{\mathrm{T}}\cdot)\bar{\hat{\psi}}\right)(t)\,dt \\ I_2 &:= \int_{\mathbb{R}^n} \sup_{(\alpha,\beta,\gamma)\in U} \alpha^{1-\frac{1}{2n}}\big|\mathcal{F}\left(\hat{\psi}(A_a S_s^{\mathrm{T}}\cdot)\bar{\hat{\psi}}\right)(t) - \mathcal{F}\left(\hat{\psi}(A_a S_s^{\mathrm{T}}\cdot)\bar{\hat{\psi}}\right)(\gamma + S_\beta A_\alpha t)\big|\,dt \\ I_3 &:= \int_{\mathbb{R}^n} \sup_{(\alpha,\beta,\gamma)\in U} \alpha^{1-\frac{1}{2n}}\big|\mathcal{F}\left(\hat{\psi}(A_a S_s^{\mathrm{T}}\cdot)\bar{\hat{\psi}}\right)(\gamma + S_\beta A_\alpha t) - \mathcal{F}\left(\hat{\psi}(A_{a\alpha} S_{\beta+\alpha^{1-1/n}s}^{\mathrm{T}}\cdot)\bar{\hat{\psi}}\right)(\gamma + S_\beta A_\alpha t)\big|\,dt. \end{aligned}$$

The integrals I_1 and I_3 can be exacly handled as in the proof of [5, Theorem 3.7]. The same holds true for I_2 except that we have to replace $t_2 \in \mathbb{R}$ in the proof [5, Theorem 3.7] by $\tilde{t} \in \mathbb{R}^{n-1}$, where $t = (t_1, \tilde{t}^{\mathrm{T}})^{\mathrm{T}}$. The corresponding estimates are detailed in [7]. □

4. Analysis of Singularities

In this section, we deal with the decay of the shearlet transform at hyperplane singularities. An $(n-m)$-dimensional hyperplane in $\mathbb{R}^n$, $1 \le m \le n$, not containing the x_1-axis can be written w.l.o.g. as

$$\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}}_{x_A} + P \underbrace{\begin{pmatrix} x_{m+1} \\ \vdots \\ x_n \end{pmatrix}}_{x_E} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad P := \begin{pmatrix} p_1^{\mathrm{T}} \\ \vdots \\ p_m^{\mathrm{T}} \end{pmatrix} \in \mathbb{R}^{m,n-m}.$$

Then we obtain for

$$\nu_m := \delta(x_A + P x_E)$$

with the Delta distribution δ that

$$\begin{aligned} \hat{\nu}_m(\omega) &= \int_{\mathbb{R}^n} \delta(x_A + P x_E) e^{-2\pi i(\langle x_A, \omega_A\rangle + \langle x_E, \omega_E\rangle)}\,dx \\ &= \int_{\mathbb{R}^{n-m}} e^{-2\pi i(-\langle P x_E, \omega_A\rangle + \langle x_E, \omega_E\rangle)}\,dx_E \\ &= \delta(\omega_E - P^{\mathrm{T}}\omega_A). \end{aligned} \tag{30}$$

The following theorem describes the decay of the shearlet transform at hyperplane singularities. We use the notation $\mathcal{SH}_\psi f(a,s,t) \sim |a|^r$ as $a \to 0$, if there exist constants $0 < c \le C < \infty$ such that

$$c|a|^r \le \mathcal{SH}_\psi f(a,s,t) \le C|a|^r \text{ as } a \to 0.$$

Theorem 4.1. *Let* $\psi \in L_2(\mathbb{R}^n)$ *be a shearlet satisfying* $\hat{\psi} \in C^\infty(\mathbb{R}^n)$. *Assume further that* $\hat{\psi}(\omega) = \hat{\psi}_1(\omega_1)\hat{\psi}_2(\tilde{\omega}/\omega_1)$, *where* $\operatorname{supp}\hat{\psi}_1 \in [-a_1, -a_0] \cup [a_0, a_1]$ *for some* $a_1 > a_0 \ge \alpha > 0$ *and* $\operatorname{supp}\hat{\psi}_2 \in Q_b$. *If*

$$(s_m, \ldots, s_{n-1}) = (-1, s_1, \ldots, s_{m-1})\,P \quad \text{and} \quad (t_1, \ldots, t_m) = -(t_{m+1}, \ldots, t_n)\,P^T,$$

then

$$\mathcal{SH}_\psi \nu_m(a,s,t) \sim |a|^{\frac{1-2m}{2n}} \quad \text{as } a \to 0. \tag{31}$$

Otherwise, the shearlet transform $\mathcal{SH}_\psi \nu_m$ *decays rapidly as* $a \to 0$.

The support condition on $\hat{\psi}_1$ and $\hat{\psi}_2$ can be relaxed toward a rapid decay of the functions.

Proof. An application of Plancherel's theorem for tempered distribution together with (30) and (7) yields

$$\begin{aligned}
\mathcal{SH}_\psi \nu_m(a,s,t) &:= \langle \nu_m, \psi_{a,s,t}\rangle \\
&= \langle \hat{\nu}_m, \hat{\psi}_{a,s,t}\rangle \\
&= \int_{\mathbb{R}^n} \delta(\omega_E - P^{\mathrm{T}}\omega_A)|a|^{1-\frac{1}{2n}} e^{2\pi i \langle t,\omega\rangle} \bar{\hat{\psi}} \left(a\omega_1, \operatorname{sgn}(a)|a|^{\frac{1}{n}}(\omega_1 s + \tilde{\omega})\right) d\omega \\
&= |a|^{1-\frac{1}{2n}} \int_{\mathbb{R}^m} e^{2\pi i \langle t_A + Pt_E, \omega_A\rangle} \bar{\hat{\psi}} \left(a\omega_1, \operatorname{sgn}(a)|a|^{\frac{1}{n}}(\omega_1 s + \begin{pmatrix} \tilde{\omega}_A \\ P^{\mathrm{T}}\omega_A \end{pmatrix})\right) d\omega_A
\end{aligned}$$

with $\tilde{\omega}_A = (\omega_2, \ldots, \omega_m)^{\mathrm{T}}$. By definition of $\hat{\psi}$ this can be rewritten as

$$\mathcal{SH}_\psi \nu_m(a,s,t) = |a|^{1-\frac{1}{2n}} \int_{\mathbb{R}^m} e^{2\pi i \langle t_A + Pt_E, \omega_A\rangle} \bar{\hat{\psi}}_1(a\omega_1) \bar{\hat{\psi}}_2 \left(|a|^{\frac{1}{n}-1}(s + \frac{1}{\omega_1}\begin{pmatrix} \tilde{\omega}_A \\ P^{\mathrm{T}}\omega_A \end{pmatrix})\right) d\omega_A.$$

Substituting $\tilde{\xi}_A = (\xi_2, \ldots, \xi_m)^{\mathrm{T}} := \tilde{\omega}_A/\omega_1$, i.e., $d\tilde{\omega}_A = |\omega_1|^{m-1}\, d\tilde{\xi}_A$, we get

$$\begin{aligned}
\mathcal{SH}_\psi \nu_m(a,s,t) &= |a|^{1-\frac{1}{2n}} \int_{\mathbb{R}} \int_{\mathbb{R}^{m-1}} e^{2\pi i \omega_1 \langle t_A + Pt_E, (1,\tilde{\xi}_A^{\mathrm{T}})^{\mathrm{T}}\rangle} \bar{\hat{\psi}}_1(a\omega_1)|\omega_1|^{m-1} \\
&\quad \times \bar{\hat{\psi}}_2 \left(|a|^{\frac{1}{n}-1}(s + \begin{pmatrix} \tilde{\xi}_A \\ P^{\mathrm{T}}(1,\tilde{\xi}_A^{\mathrm{T}})^{\mathrm{T}} \end{pmatrix})\right) d\tilde{\xi}_A d\omega_1
\end{aligned}$$

and further by substituting $\xi_1 := a\omega_1$

$$\begin{aligned}
\mathcal{SH}_\psi \nu_m(a,s,t) &= |a|^{1-m-\frac{1}{2n}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} e^{2\pi i \frac{\xi_1}{a} \langle t_A + Pt_E, (1,\tilde{\xi}_A^{\mathrm{T}})^{\mathrm{T}}\rangle} |\xi_1|^{m-1} \bar{\hat{\psi}}_1(\xi_1)\, d\xi_1 \\
&\quad \times \bar{\hat{\psi}}_2 \left(|a|^{\frac{1}{n}-1}(s + \begin{pmatrix} \tilde{\xi}_A \\ P^{\mathrm{T}}(1,\tilde{\xi}_A^{\mathrm{T}})^{\mathrm{T}} \end{pmatrix})\right) d\tilde{\xi}_A.
\end{aligned}$$

Finally, by substituting $\tilde{\omega}_A := |a|^{\frac{1}{n}-1}(\tilde{\xi}_A + s_a)$, where $s_a := (s_1, \ldots, s_{m-1})^{\mathrm{T}}$ and $s_e := (s_m, \ldots, s_{n-1})^{\mathrm{T}}$, we obtain

$$\begin{aligned}
\mathcal{SH}_\psi \nu_m(a,s,t) &= |a|^{\frac{1-2m}{2n}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} e^{2\pi i \frac{\xi_1}{a} \langle t_A + Pt_E, (1, |a|^{1-1/n}\tilde{\omega}_A^{\mathrm{T}} - s_a^{\mathrm{T}})\rangle} |\xi_1|^{m-1} \bar{\hat{\psi}}_1(\xi_1)\, d\xi_1 \\
&\quad \times \bar{\hat{\psi}}_2 \begin{pmatrix} \tilde{w}_A \\ |a|^{\frac{1}{n}-1}(s_e - P^{\mathrm{T}}\begin{pmatrix} -1 \\ s_a \end{pmatrix}) + P^{\mathrm{T}}\begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix} \end{pmatrix} d\tilde{\omega}_A.
\end{aligned}$$

If the vector

$$s_e - P^{\mathrm{T}}\begin{pmatrix} -1 \\ s_a \end{pmatrix} \neq 0_{n-m} \tag{32}$$

then at least one component of its product with $|a|^{1/n-1}$ becomes arbitrary large as $a \to 0$. On the other hand, by the support property of $\hat{\psi}_2$, we conclude that $\hat{\psi}_2(\tilde{\omega}_A, \cdot)$ becomes zero if we $\tilde{\omega}_A$ is not in $Q_{(b_1,\ldots,b_{m-1})} \subset R^{m-1}$. But for all $\tilde{\omega}_A \in Q_{(b_1,\ldots,b_{m-1})}$ at least one component of

$$|a|^{\frac{1}{n}-1}(s_e - P^{\mathrm{T}}\begin{pmatrix} 1 \\ s_a \end{pmatrix}) + P^{\mathrm{T}}\begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix}$$

is not within the support of $\hat{\psi}_2$ for a sufficiently small so that $\hat{\psi}_2$ becomes zero again. Assume now that we have equality in (32). Then

$$\begin{aligned}
\mathcal{SH}_\psi \nu_m(a,s,t) &= |a|^{\frac{1-2m}{2n}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} e^{2\pi i \frac{\xi_1}{a} \langle t_A + Pt_E, (1, |a|^{1-1/n} \tilde{\omega}_A^{\mathrm{T}} - s_a^{\mathrm{T}})\rangle} |\xi_1|^{m-1} \bar{\hat{\psi}}_1(\xi_1)\, d\xi_1 \\
&\quad \times \bar{\hat{\psi}}_2 \begin{pmatrix} \tilde{w}_A \\ P^{\mathrm{T}} \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix} \end{pmatrix} d\tilde{\omega}_A \\
&= C\,|a|^{\frac{1-2m}{2n}} \int_{\mathbb{R}^{m-1}} \tilde{\psi}_1^{(m-1)} \Big(\langle t_A + Pt_E, (1, |a|^{1-1/n} \tilde{\omega}_A^{\mathrm{T}} - s_a^{\mathrm{T}})\rangle / a \Big) \\
&\quad \times \bar{\hat{\psi}}_2 \begin{pmatrix} \tilde{w}_A \\ P^{\mathrm{T}} \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix} \end{pmatrix} d\tilde{\omega}_A \\
&= C\,|a|^{\frac{1-2m}{2n}} \int_{\mathbb{R}^{m-1}} \tilde{\psi}_1^{(m-1)} \left(\langle t_A + Pt_E, \begin{pmatrix} |a|^{1/n-1} \\ \tilde{\omega}_A^{\mathrm{T}} - |a|^{1/n-1} s_a \end{pmatrix} \rangle\, |a|^{-1/n} \right) \\
&\quad \times \bar{\hat{\psi}}_2 \begin{pmatrix} \tilde{w}_A \\ P^{\mathrm{T}} \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix} \end{pmatrix} d\tilde{\omega}_A,
\end{aligned}$$

where $\tilde{\psi}_1$ has the Fourier transform $\hat{\tilde{\psi}}_1(\xi_1) := \bar{\hat{\psi}}_1(\xi_1)$ for $\xi_1 \geq 0$ and $\hat{\tilde{\psi}}_1(\xi_1) := -\bar{\hat{\psi}}_1(\xi_1)$ for $\xi_1 < 0$. Since by our assumptions the support of $\hat{\psi}_1$ is bounded away from the origin, we see that $\hat{\tilde{\psi}}_1$ is again in $C^\infty(\mathbb{R})$. If $t_A + Pt_E \neq 0_m$, then, since $\hat{\psi}_1 \in C^\infty$ the first function decays rapidly as $a \to 0$ for all $\tilde{w}_A$ in the bounded domain, where $\hat{\psi}_2$ doesn't become zero. Consequently, the value of the shearlet transform decays rapidly. If $t_A + Pt_E = 0_m$, then

$$\mathcal{SH}_\psi \nu_m(a,s,t) \;=\; C\,|a|^{\frac{1-2m}{2n}}\, \tilde{\psi}_1^{(m-1)}(0) \int_{\mathbb{R}^{m-1}} \bar{\hat{\psi}}_2 \begin{pmatrix} \tilde{w}_A \\ P^{\mathrm{T}} \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix} \end{pmatrix} d\tilde{\omega}_A \;\sim\; |a|^{\frac{1-2m}{2n}}.$$

This finishes the proof. □

Remark 4.2. Other choices of the dilation matrix are possible, e.g.,

$$A_a := \begin{pmatrix} a & 0_{n-1}^{\mathrm{T}} \\ 0_{n-1} & \operatorname{sgn}(a)\sqrt{|a|}\, I_{n-1} \end{pmatrix}.$$

Then we have to replace (31) by $|a|^{\frac{n-2m-1}{4}}$ which increases for $n < 2m+1$ as $a \to 0$. Therefore, we prefer our choice.

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