Frames, Sparsity and Nonlinear Inverse Problems*

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Summary

This work is concerned with nonlinear inverse problems where the solution is assumed to have a sparse expansion with respect to several preassigned bases or frames. We develop a scheme which allows to minimize a Tikhonov functional where the usual quadratic regularization term is replaced by one–homogeneous (typically weighted ℓ_p, $1 \leq p \leq 2$) penalties on the coefficients (or isometrically transformed coefficients) of such multi–frame expansions. The computation of the solution amounts in this setting to a system of Landweber–fixed–point iterations with thresholding applied in each fixed–point iteration step.

Extended Outline

We consider the computation of an approximation to a solution of a nonlinear operator equation

$$T(x) = y \ , \tag{1}$$

where $T : X \to Y$ is an operator between Hilbert spaces X, Y. In case of having only noisy data y^δ with $\|y^\delta - y\| \leq \delta$ available, there might be the problem of ill-posedness (in the sense of a discontinuous dependency of the solution on the data). Thus problem (1) has to be stabilized by regularization methods. In recent years, many of the well known methods for linear inverse problems have been generalized to nonlinear operator equations. But so far all the proposed schemes for nonlinear problems incorporate at most quadratic regularization whereas in many applications the solution is assumed to have sparse expansion which immediately leads to the involvement of nonquadratic penalties, e.g. ℓ_p norms with $p < 2$. In linear lore, this problem is still solved, see [2]. In nonlinear inverse problems there is an approach, see [5], which solves nonlinear operator equations with sparsity constraints. However, recent developments indicate that (higly) redundant systems, such as frames or systems of frames may yield a gain in this context (optimal representation/decomposition of the solution to be reconstructed). When dealing with dictionaries of frame systems, there exist certain methods, e.g. such as basis pursuit [1], that allow a decomposition of signals/functions into an optimal superposition of dictionary elements, where optimal means having smallest ℓ_1 norm of coefficients among all such decompositions. In [8], we have presented a method which combines an iterated thresholding scheme for solving linear inverse problems while requiring that the solution is assumed to have a sparse expansion in a multi–frame dictionary. In this paper, we also assume that

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the solution has a sparse expansion in a multi–frame dictionary but we aim now to extend the theory to nonlinear inverse problems with mixed multi–sparsity constraints. Thus the main result of this paper, coming out by combing results and technologies elaborated in [3, 4], [6, 5], and [8], is the development of a new method which is sort of thresholding Landweber iteration for solving a system of fixed point equations. This scheme is numerically illustrated by solving a few image processing task, but we also provide a regularization result which shows that this method is also well suited for ill-posed problems.

As in [8], let us assume we are given a finite family of preassigned frames $\{\phi^i_\lambda\}_{\lambda\in\Lambda_i, i\in\mathcal{I}} \subset X$, $n = \mathrm{card}(\mathcal{I})$, for which we have associated frame operators

$$F_i : X \to \ell_2 \text{ via } F_i x = \{\langle x, \phi^i_\lambda\rangle\}_{\lambda\in\Lambda_i} \text{ with } A_i \cdot I \leq F_i^* F_i \leq B_i \cdot I \ .$$

The variational formulation of the nonlinear inverse problem in a multi–frame setting with so–called multi–sparsity, or more general, multi–one–homogeneous constraints can be now casted as follows: find sequences of coefficients $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n) \in (\ell_2)^n$ such that

$$J_\alpha(\boldsymbol{g}) = \|y^\delta - T(K\boldsymbol{g})\|_Y^2 + 2\alpha \cdot \Psi_{\boldsymbol{L}}(\boldsymbol{g}) \tag{2}$$

is minimized, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\Psi_{\boldsymbol{L}}(\boldsymbol{g}) = (\Psi_1(\boldsymbol{L}_1\boldsymbol{g}_1), \ldots, \Psi_n(\boldsymbol{L}_n\boldsymbol{g}_n))$. In our case, $K\boldsymbol{g} = K(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n) = \sum_{i\in\mathcal{I}} F_i^*\boldsymbol{g}_i$, but one could also involve, as in [8], additional linear and bounded operators E_i, i.e. $K_E(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n) = \sum_{i\in\mathcal{I}} E_i F_i^* \boldsymbol{g}_i$. Moreover, the Ψ_i stand for positive, one–homogeneuos, lower semi–continuous and convex penalties (which are usually some weighted ℓ_p norms of the frame coefficients), and the infinite matrices $\boldsymbol{L}_i$ are restricted to be isometric mappings. In particular, we also need to require,

$$\|\boldsymbol{g}\|_{(\ell_2)^n} \leq \|\Psi_{\boldsymbol{L}}(\boldsymbol{g})\|_{\ell_1} \ . \tag{3}$$

The main strategies developed in [6, 5] seem also to be adequate for minimizing (2), i.e. when dealing with multi–sparsity, or more general, with multi–one–homogeneous constraints.

The general idea for solving the nonlinear inverse problem in a multi–frame setting goes thus as follows: we replace (2) by a sequence of functionals from which we hope that they are easier to treat and that the sequence of minimizers converge in some sense to, at least, a critical point of (2). To be more concrete, for $\boldsymbol{g} \in (\ell_2)^n$ and some auxiliary $\boldsymbol{a} \in (\ell_2)^n$, we introduce

$$J^s_\alpha(\boldsymbol{g}, \boldsymbol{a}) := J_\alpha(\boldsymbol{g}) + C\|\boldsymbol{g} - \boldsymbol{a}\|^2_{(\ell_2)^n} - \|T(K\boldsymbol{g}) - T(K\boldsymbol{a})\|^2_Y \tag{4}$$

and create an iteration process by:

1. Pick $\boldsymbol{g}_0 \in (\ell_2)^n$ and some proper constant $C > 0$
2. Derive a sequence $\{\boldsymbol{g}_k\}_{k=0,1,\ldots}$ by the iteration:

$$\boldsymbol{g}_{k+1} = \arg \min_{\boldsymbol{g}_k\in(\ell_2)^n} J^s_\alpha(\boldsymbol{g}, \boldsymbol{g}_k) \qquad k = 0, 1, 2, \ldots$$

It can be shown that, in order to prove norm convergence of the iterates $\boldsymbol{g}_k$ towards a critical point of J_α, we have to restrict ourselves to a class of nonlinear problems for which all of the following three requirements hold true,

$$\begin{aligned} &\boldsymbol{g}_k \stackrel{w}{\to} \boldsymbol{g} \Longrightarrow T(K\boldsymbol{g}_k) \to T(K\boldsymbol{g}) \quad , \\ &F_j T'(K\boldsymbol{g}_k)^* z \to F_j T'(K\boldsymbol{g})^* z \quad \text{, for all } z \text{ and } j \quad , \\ &\|T'(K\boldsymbol{g}) - T'(K\boldsymbol{g}')\| \leq LB\|\boldsymbol{g} - \boldsymbol{g}'\|_{(\ell_2)^n} \quad . \end{aligned} \tag{5}$$

In the talk we shall explain how the replacement functionals are constructed and we shall discuss the well–posedness of the resulting problem. Moreover, we derive conditions on the minimizing elements. As the main result, we show strong convergence of the iterates towards a critical point and we state conditions for which we may ensure that the scheme is indeed a regularization scheme. We end with demonstrating the capabilities of the proposed scheme by solving nonlinear image processing tasks. An extended consideration of this topic can be found in [7].

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