The Canonical Coherent States Associated With Quotients of the Affine Weyl-Heisenberg Group*

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Abstract

This paper is concerned with the uncertainty principle in the context of the affine-Weyl-Heisenberg group in one and two dimensions. As the representation of this group fails to be square integrable, we explore various admissible sections of this group, and calculate the resulting uncertainty principles as well as its minimizers with respect to these sections. Previous studies have shown that these sections give rise to mixed smoothness spaces. We demonstrate that the minimizers obtained for these sections actually interpolate between Gabor and wavelets functions.

Keywords: Affine Weyl-Heisenberg group, representations via quotients, uncertainty principles, minimizing states, coorbit spaces.
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1 Introduction

Several applications in the fields of signal and image processing involve the convolution of some filter bank with the signal to be processed. This convolution provides local features of the signal or image. Usually, we take some mother function and generate the whole filter bank using group operations, such as translations, rotations, scaling and modulations. Determining the mother function is usually done by some ad-hoc methods that account for the specific application involved. Special attention was given in the past to those functions that provide the maximal accuracy, and hence minimal uncertainty, for the values of the features involved, e.g. translation, scaling, rotation etc.

One classical example consists of the uncertainty relation associated with the Short Time Fourier Transform (STFT). The STFT or so-called Gabor transform, see [6], is obtained by applying the action of the Weyl-Heisenberg group to a suitable window function and taking the inner product with the signal. Moreover, it is a classical result that choosing the Gaussian function as the window function minimizes the uncertainty relation and therefore gives rise to canonical coherent states of the Weyl-Heisenberg group.

More recent studies considered the uncertainty principles which are related to the affine group in one dimension and the similitude group as well as the affine group in two dimensions [1, 3, 9]. For the one dimensional affine group it was possible to find an analytical solution of the form:

$$\psi(x) = c(x-\eta)^{-\frac{1}{2}-i\eta\mu_2+i\mu_1}, \tag{1}$$

where c is some constant, η is purely imaginary and $\mu_1, \mu_2 \in \mathbb{R}$. However, for the two dimensional case, it was not possible to find solutions which simultaneously minimize the combined uncertainty with respect to all the parameters involved, and therefore solutions that accounted for various subgroups were employed.

In this study we focus on the affine Weyl-Heisenberg group. There is a growing interest in this group as well as in the integral transforms associated with it, and several studies have already dealt with it [4, 9, 12, 13, 14, 15].

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The results of [4], where some mixed forms of smoothness spaces that lie in between Besov and modulation spaces have been constructed, are of particular interest for the results of the present paper. These mixed smoothness spaces are the α-modulation spaces, see [4]. In addition, the results of [13], where interpolating wavelet packets between the Gabor and wavelet transform are generated, did influence our approach. It was already shown, see [14], that the representation of this group fails to be square integrable. A possible remedy is to factor out a suitable closed subgroup and work with quotients. We follow this approach in this paper and calculate the minimizers for some subgroups. The results are extended to the two dimensional affine Weyl-Heisenberg group.

This paper is organized as follows: First, we discuss some basic results and summarize related work. Next, we calculate the minimizers for the one dimensional affine Weyl-Heisenberg group and address the issue of using admissible sections. Finally, we continue by analyzing the two-dimensional affine Weyl-Heisenberg group and explore some possible subgroups for obtaining valid minimizers.

2 Background and Related Work

A general theorem which is well-known in quantum mechanics and harmonic analysis [5] relates an uncertainty principle to any two self-adjoint operators and provides a mechanism for deriving a minimizing function for the uncertainty relation. Before repeating this well-known result on uncertainties, let us fix some notation. Let A, B be two self-adjoint operators. Their *commutator* is defined by

$$[A, B] := AB - BA,$$

the *expectation* of A with respect to some state $\psi \in \mathrm{dom}(A)$ by

$$\mu(A) := \mu_A := \langle A\psi, \psi\rangle$$

and, finally, the *variance* of A with respect to some state $\psi \in \mathrm{dom}(A)$ by

$$\Delta A_\psi := \mu((A - \mu(A))^2).$$

Theorem 1 *Given two self-adjoint operators A and B, then for all $\psi \in \mathrm{dom}(A) \cap \mathrm{dom}(B)$ they obey the uncertainty relation:*

$$\Delta A_\psi \Delta B_\psi \;\geq \frac{1}{2}|\langle [A, B]\rangle|. \tag{2}$$

A state ψ is said to have minimal uncertainty *if the above inequality turns into an equality. This happens iff there exists an $\eta \in i\mathbb{R}$ such that*

$$(A - \mu_A)\psi = \eta(B - \mu_B)\psi. \tag{3}$$

The Weyl-Heisenberg group as well as the affine group are both related to well-known transforms in signal processing: the STFT and the wavelet transform, respectively. Both can be derived from square integrable representations of these groups. The windowed Fourier transform is related to the Weyl-Heisenberg group and the wavelet transform is related to the affine group. The linearized operation of the group at the identity element can be described by the infinitesimal generators of the related Lie algebra. If the group representation is unitary, then the infinitesimal generators can transformed to be self-adjoint operators. Thus, the general uncertainty theorem stated above provides a tool for obtaining uncertainty principles using these infinitesimal generators. In the case of the Weyl-Heisenberg group, the canonical functions that minimize the corresponding uncertainty relation are Gaussian functions.

The canonical functions that minimize the uncertainty relations for the affine group in one dimension and for the similitude group in two dimensions, were the subject of previous studies [1, 3]. In these studies, it was shown that there is no non-trivial canonical function that minimizes the uncertainty relation associated with the similitude group of $\mathbb{R}^2$, $SIM(2)$. Thus, there is no non-zero solution for the set of differential equations obtained for these group generators. Rather than using the original generators of the $SIM(2)$ group, a different set of operators was used in [3] that includes elements of the enveloping algebra, i.e., polynomials in the generators of the algebra, in order to obtain the $2D$ isotropic Mexican hat as a minimizer. Further results were achieved in [1] where a symmetry in the set of commutators was obtained for the $SIM(2)$ group and a possible minimizer in the frequency domain for some fixed direction was obtained. This solution is a real valued wavelet which is confined to some convex cone in the positive half-plane of the frequency space with an exponential decay inside the cone.

The extension of these studies to the affine group in two dimensions resulted in two possible solutions [9]. The first accounted for the overall scaling and rotation and utilizes the results of [1, 3]. The second solution was obtained by exploiting a symmetry in the group of commutators which led to

$$\psi(x,y) = (\eta + x)^{-\frac{1}{2} - i\mu_{11} + i\eta\mu_{bx}} e^{i\mu_{b_y} y}. \tag{4}$$

It is square integrable with respect to the variable x if we select $\mathrm{Re}(\eta)b_x \geq 0$, but not square integrable in terms of the variable y, although it is periodic.

The affine Weyl-Heisenberg (AWH) group has already been addressed in this context in the early $90's$. The paper [14] considered wavelets associated with representations of the AWH group. It shows that the canonical representation of the AWH group is not square integrable, but can be regularized with some density function. This work was later extended to N-dimensional AWH wavelets [15]. In [10] a scaling was introduced in the Heisenberg group with an intertwining operator. More recently, [12] proposed a mechanism to construct generalized uncertainty principles and their minimizing wavelets in anisotropic Sobolev spaces. A new set of uncertainty principles was introduced in this paper by weakening the two operator relations and by introducing a multi-dimensional operator setting. Recently, a study [4] has considered generalizations of the coorbit space theory based on group representations modulo quotients. This is based on applying the general theory to the AWH group and obtaining families of smoothness spaces that can be identified with the α-modulation spaces.

3 The $1D$ Affine Weyl-Heisenberg Group

The affine Weyl-Heisenberg group is generated by time translations $b \in \mathbb{R}$, frequency translations $\omega \in \mathbb{R}$, spatial dilations $a \in \mathbb{R}_+$, and a toral component $\phi \in \mathbb{R}$, and is equipped with the group law

$$(b, \omega, a, \phi) \circ (b', \omega', a', \phi') = (b + ab', \omega + \omega'/a, aa', \phi + \phi' + \omega b' a).$$

The AWH group can be viewed as the extension of the affine group, incorporating frequency translations or, alternatively, as the extension of the Weyl-Heisenberg group incorporating dilations. The Stone-von-Neumann representation of G_{AWH} on $L_2(\mathbb{R})$ is given by:

$$[U(b, \omega, a, \phi)\psi](x) = a^{-\frac{1}{2}} e^{i\omega(x-b)} e^{i\phi} \psi(\frac{x-b}{a}). \tag{5}$$

This representation, however, fails to be square integrable [13]. The AWH group raises a special interest as it "contains" both, the affine group as well as the Weyl-Heisenberg group: If we consider cases where $a = 1$, we are in the Weyl-Heisenberg framework, and if we consider cases where $\omega = 0$ we are in the affine framework. Two independent studies have regarded these attributes, and suggested a specific section of the AWH [4, 13], where the scale is represented as a function of the frequency. It was proven that this section is admissible. In addition, this paper introduces a mechanism that starts at the Weyl-Heisenberg case and allows a smooth transition towards the affine case.

In what follows, we regard this section, and calculate the appropriate minimizing functions with respect to the uncertainty principle related to it. Then, we study the section where the frequency is regarded as a function of the scale, consider its admissibility and calculate the appropriate minimizers.

3.1 The Sections Where the Scale is a Function of the Frequency

These kinds of sections appear in a quite natural way in the context of α-modulation spaces. In the α-modulation spaces framework it is desired to construct mixed forms of smoothness spaces that lie in between Besov spaces (related to the affine group) and modulation spaces (related to the Weyl-Heisenberg group). For that, a group that contains all the components of Besov and modulation spaces is required, such as the affine Weyl-Heisenberg group. As the representation of this group is not square integrable, it is suggested to factor out a closed subgroup and work with the quotients.

We will consider G_{AWH}/H with

$$H := (0, 0, a, \phi) \in G_{AWH}$$

and Borel sections that do not depend on b, namely:

$$\sigma(b, \omega) = (b, \omega, \beta(\omega), 0).$$

Further, it was shown in [4] that the specific section

$$\beta(\omega) = \eta_\alpha(\omega)^{-1} = (1 + |\omega|)^{-\alpha}$$

is admissible.

Next, we consider the effect of varying the value of $\alpha \in [0, 1]$. If $\alpha = 0$ then we obtain:

$$\beta(\omega) = \eta_0(\omega)^{-1} = (1 + |\omega|)^0 = 1.$$

Thus, there are practically no dilations and we obtain Gabor analysis. For $\alpha \to 1$ we obtain:

$$\beta(\omega) = \eta_\alpha(\omega)^{-1} = (1 + |\omega|)^{-\alpha} \overset{|\alpha| \to 1}{\longrightarrow} \frac{1}{1 + |\omega|}.$$

Thus, the frequency translations and modulations are inversely proportional which is close to wavelet analysis. The intermediate case for which $\alpha = \frac{1}{2}$ is known as the Fourier-Bros-Iagolnitzer transform.

The representation for the quotient as a function of α is then given by

$$[U(b, \omega, \eta_\alpha^{-1}(\omega))\psi](x) = (1 + |\omega|)^{\frac{\alpha}{2}} e^{i\omega(x-b)} \psi\left((1 + |\omega|)^\alpha (x - b)\right).$$

As can be seen, this representation is not C^1 for $\omega = 0$. Nevertheless, when calculating the infinitesimal generators, we may take the one-sided derivatives.

Lemma 2 *The infinitesimal operators T_b, T_ω associated with the one dimensional G_{AWH} are given by*

$$(T_b\psi)(x) = -i\frac{\partial}{\partial x}\psi(x), \quad \textit{and} \quad (T_\omega\psi)(x) = (i\frac{\alpha}{2} - x)\psi(x) + i\alpha x\frac{\partial}{\partial x}\psi(x). \tag{6}$$

The state ψ which is the minimizer of the associated uncertainty is of the form

$$\psi(x) = e^{\frac{-ix}{\alpha}}(\alpha\lambda x + 1)^{-\frac{1}{2} - \frac{i\mu_\omega}{\alpha} + \frac{i\mu_b}{\alpha\lambda} + \frac{i}{\alpha^2\lambda}}. \tag{7}$$

Proof; Taking the (one-sided) derivatives with respect to ω and b and evaluating them at $b = 0, \omega = 0$ leads to

$$\frac{\partial}{\partial b}U(b, \omega, \eta_\alpha(\omega), 0)\psi|_{b=0,\omega=0}(x) = -\frac{\partial}{\partial x}\psi(x), \qquad \frac{\partial}{\partial \omega}U(b, \omega, \eta_\alpha(\omega), 0)\psi|_{b=0,\omega=0}(x) = (\frac{\alpha}{2} + ix)\psi(x) + \alpha x\frac{\partial}{\partial x}\psi(x).$$

These operators are not self-adjoint, but multiplication with the imaginary unit i yields self-adjoint operators $T_b = i\,\frac{\partial}{\partial b}U$ and $T_\omega = i\,\frac{\partial}{\partial \omega}U$ in the Lie algebra under consideration. This proves (6).

The commutator between these two operators is non-zero. This implies that we cannot exactly measure the mean values of the spatial frequency and the position simultaneously. By means of Theorem 1, we may calculate those states that minimize the corresponding uncertainty principle. Indeed, eq. (3) provides us with the differential equation

$$-i\frac{\partial}{\partial x}\psi(x) - \mu_b\psi(x) = \lambda\left((\frac{i\alpha}{2} - x)\psi(x) + i\alpha x\frac{\partial}{\partial x}\psi(x) - \mu_\omega\psi(x)\right),$$

i.e.,

$$\frac{\partial}{\partial x}\psi(x) = i\psi(x)\left(\frac{-\lambda x + \frac{i\lambda\alpha}{2} - \lambda\mu_\omega + \mu_b}{\alpha\lambda x + 1}\right). \tag{8}$$

Now (8) can be solved by separation of variables which leads us to (7). □

In order for this solution to be square integrable, the following asumptions have to be met: suppose that $\lambda = i\gamma$ where $\gamma \in \mathbb{R}$. If $\gamma < 0$, resp. if $\gamma > 0$, then the solution is square integrable if $\mu_b > -\frac{1}{\alpha}$, resp. if $\mu_b < -\frac{1}{\alpha}$. In Figure 1 we have plotted ψ, which is the minimizer for this section of the AWH group in $1D$.

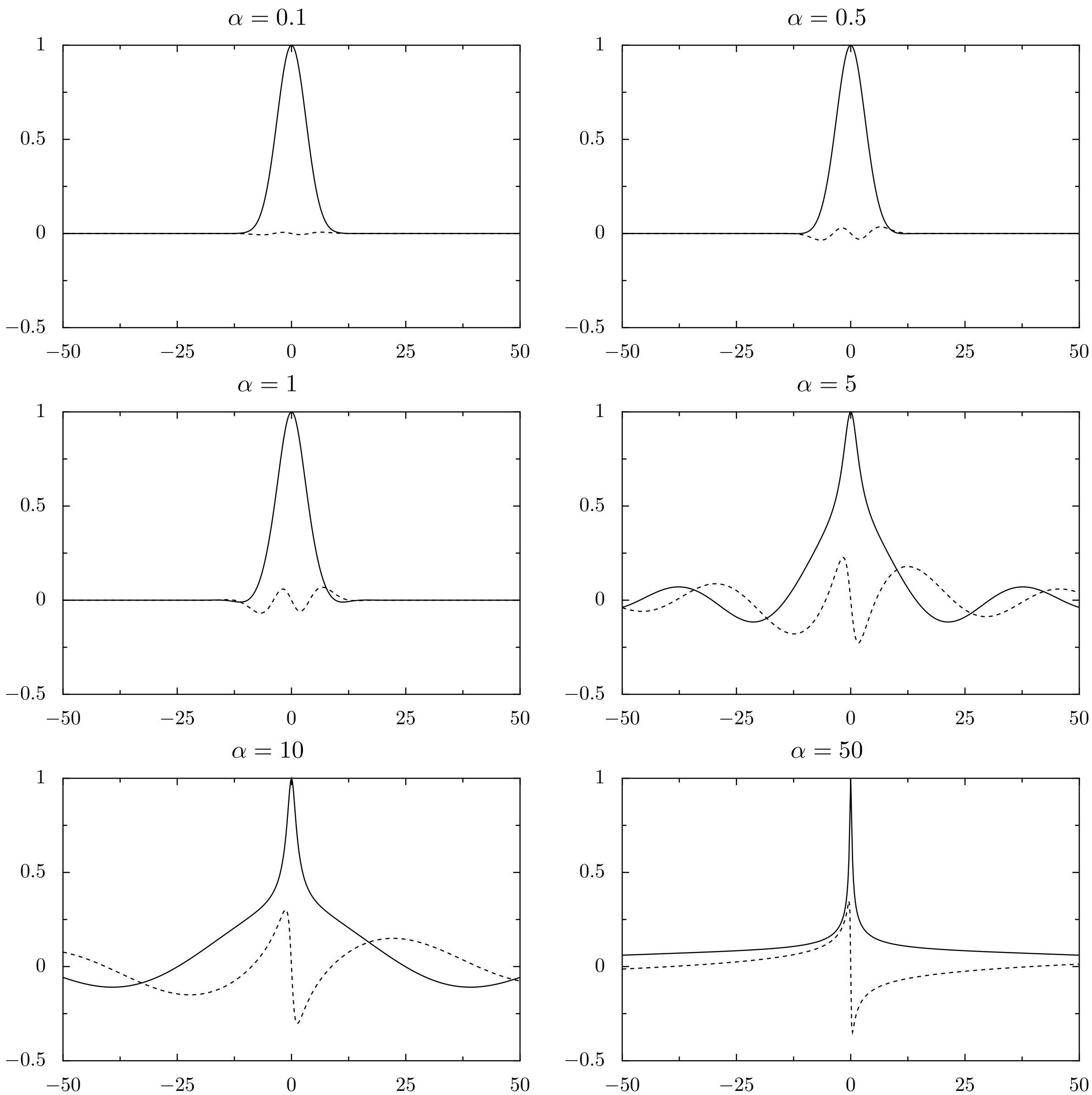

Figure 1: The minimizers of the AWH uncertainty for $\lambda = 0.1i$, $\mu_b = \mu_\omega = 0$ and different values of α. The real part is plotted solid, the imaginary part is dashed. Note the transition from a Gaussion for small α (Weyl-Heisenberg case) to the Cauchy wavelet for large α (affine case).

3.2 The Sections that Regard the Frequency as a Function of the Scale

In the previous section we have considered the section

$$\beta(\omega) = \eta_\alpha(\omega)^{-1} = (1 + |\omega|)^{-\alpha}.$$

We note that it is not possible to obtain the affine framework, which would be related to $\omega \equiv 0$ using this approach. Hence, let us explore the inverse relationship

$$\omega = \zeta_\alpha(a) = a^{-\frac{1}{\alpha}} - 1 \ ,$$

which determines the frequency as a function of the scale a.

Let us denote $\kappa = \frac{1}{\alpha}$, and restrict the discussion to values of κ ranging between 0 and 1 (corresponding to values of α ranging between 1 and ∞). Thus we obtain $\omega = \zeta(a) = a^{-\kappa} - 1$. If κ is selected to be zero, we then obtain no frequency modulation as then $\omega = 0$, and thus we are in the affine case. If κ is selected to be one, we again observe reciprocal relations between scale and frequency of the form: $|\omega| = a^{-1} - 1$, which is the same as $a = \frac{1}{1+|\omega|}$, and corresponds to the case of Gabor-like wavelets.

This concept has again an interpretation in the group theoretical setting. Once more we are working with the affine Weyl-Heisenberg group G_{AWH}, but this time we consider the subgroup

$$H := (0, \omega, 1, \phi) \in G_{AWH}$$

and the associated quotient group $X = G_{AWH}/H$. In order to make this setting well-defined, first of all it is necessary to establish square-integrability, see, e.g., [1] for a detailed discussion. In general, let a quasi-invariant measure μ on X and a section σ be given. Then a unitary representation U of G on a Hilbert space $\mathcal{H}$ is called *square-integrable modulo* $(H; \sigma)$ if there exists a function $\psi \in \mathcal{H}$ such that the self-adjoint operator $A_\sigma : \mathcal{H} \to \mathcal{H}$ (depending on σ and ψ) weakly defined by

$$A_\sigma f := \int_X \langle f, U(\sigma(h))\psi\rangle_{\mathcal{H}} U(\sigma(x))\psi d\mu(h) \tag{9}$$

is bounded and has a bounded inverse. The function ψ is then called *admissible.* If A_σ is a multiple of the identity then we are in the strictly admissible case. We consider the case of the affine group in the following lemma. Note, that in the lemma and only in this lemma, we replace ω by $2\pi\omega$ in the definition of the representation $U(a, b, \omega, \phi)\psi$.

Lemma 3 *Let $\psi \in L_2(\mathbb{R})$. The operator A_σ in* (9) *for the affine Weyl-Heisenberg group, i.e.*

$$A_\sigma f(x) = \int\int \langle f, \psi_{a,b}\rangle \psi_{a,b}(x) \ db \ \frac{da}{a} \tag{10}$$

with the section $\sigma(a, b) = (b, \zeta(a), a, 0)$, i.e.,

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} e^{2\pi i \zeta(a)(x-b)} \psi \ \left(\frac{x-b}{a}\right), \tag{11}$$

can be written as a Fourier multiplier operator:

$$\widehat{A_\sigma f} = m_\zeta \hat{f} \tag{12}$$

with the symbol

$$m_\zeta(\gamma) := \int_{\mathbb{R}} |\hat{\psi}(a(\gamma - \zeta(a)))|^2 da. \tag{13}$$

Proof. We follow the approach of [8]. For the sake of this calculation we use the approximation

$$A^T_\sigma f(x) = \int\int \langle f, \psi_{a,b}\rangle \psi_{a,b}(x) \ \chi_{[-\frac{T}{2},\frac{T}{2}]}(b) \ db \frac{da}{a}. \tag{14}$$

In order to compute $\widehat{A_\sigma^T f(\gamma)}$, we first derive the Fourier transform of $\psi_{a,b}$:

$$\hat{\psi}_{a,b}(\gamma) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{2\pi i \zeta(a)(x-b)} e^{-2\pi i \gamma x} \psi\left(\frac{x-b}{a}\right) dx. \tag{15}$$

If we apply the change of variables $y = \frac{x-b}{a}$ we obtain

$$\hat{\psi}_{a,b}(\gamma) \;=\; \sqrt{a} e^{-2\pi i b\gamma} \hat{\psi}(a(\gamma - \zeta(a))). \tag{16}$$

With the help of Plancherel's theorem we further obtain

$$\langle f, \psi_{a,b} \rangle = \langle \hat{f}, \hat{\psi}_{a,b} \rangle = \int \sqrt{a} \hat{f}(\omega) e^{2\pi i b\omega} \bar{\hat{\psi}}(a(\omega - \zeta(a))) d\omega \tag{17}$$

and thus

$$\begin{aligned}
\widehat{A_\sigma^T f(\gamma)} &= \int\int \hat{f}(\omega) \bar{\hat{\psi}}(a(\omega - \zeta(a))) \hat{\psi}(a(\gamma - \zeta(a))) \int e^{-2\pi i b(\gamma-\omega)} \chi_{[-\frac{T}{2},\frac{T}{2}]}(b)\; db da d\omega \\
&= \int \hat{\psi}(a(\gamma - \zeta(a))) \int \hat{f}(\omega) \bar{\hat{\psi}}(a(\omega - \zeta(a))) T \frac{\sin(\pi(\gamma-\omega)T)}{\pi(\gamma-\omega)T} d\omega da.
\end{aligned}$$

The term

$$T \frac{\sin(\pi(\gamma-\omega)T)}{\pi(\gamma-\omega)T} \tag{18}$$

can be seen as an approximation of a δ-function when T approaches infinity. Thus, we obtain

$$\widehat{A_\sigma(f)}(\gamma) = \hat{f}(\gamma) \int |\hat{\psi}(a(\gamma - \zeta(a)))|^2 da = \hat{f}(\gamma) m_\zeta(\gamma). \tag{19}$$

$\square$

We can now determine whether A_σ is bounded with a bounded inverse as this is true if and only if

$$C_1 \leq m_\zeta(\gamma) \leq C_2 \tag{20}$$

almost everywhere for constants $0 < C_1, C_2 < \infty$. If we check the admissibility of the section for the case $\kappa = 1$, i.e. $\zeta(a) = \frac{1}{a} - 1$, then

$$\begin{aligned}
m_\zeta(\gamma) &= \int_0^\infty |\hat{\psi}(a(\gamma - \frac{1}{a} + 1))|^2 da \\
&= \int_0^\infty |\hat{\psi}(a(\gamma + 1) - 1))|^2 da \\
&= \frac{1}{|\gamma+1|} \int_0^\infty |\hat{\psi}(x)|^2 dx \qquad \stackrel{|\gamma|\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Thus, this shows that ζ is not admissible even for this easy case.

Nevertheless, a possible remedy for the non-admissibility of this section can be found in the framework of quasi-coherent states [2]. In this case, although $m_\zeta(\gamma)$ calculated above does not satisfy (20), the corresponding integral with respect to a positive density $\iota(a, \gamma) = \frac{1}{a}$ is bounded from above and below

$$\tilde{m}_\zeta(\gamma) \;=\; \int_0^\infty |\hat{\psi}(a(\gamma - \frac{1}{a} + 1))|^2 \iota(a,\gamma)\; da \;=\; \int_0^\infty \frac{|\hat{\psi}(x)|^2}{x}\; dx\;.$$

This reproduces the classical admissibility conditions for the continuous wavelet transform and can be derived from Lemma 3 by using the integration measure $db\; da/a^2$.

This leads to quasi-coherent states that have the standard properties of the covariant coherent states: overcompleteness, resolution of a positive operator A_σ and having a reproducible kernel.

We now turn to calculating the infinitesimal operators and the wavelets minimizing the corresponding uncertainty relation. The choice of the section $\omega = \zeta(a)$ leads to the following represenatation:

$$[U(b, \zeta(a), a, 0)\ \psi](x) = \frac{1}{\sqrt{a}} e^{i(a^{-\kappa}-1)(x-b)} \psi\left(\frac{x-b}{a}\right).$$

Next, we calculate the two infinitesimal generators related to this representation.

Lemma 4 *The infinitesimal operators with respect to the representation above are*

$$T_a\psi(x) = (\kappa x - \frac{i}{2})\psi(x) - ix\psi_x(x) \quad \textit{and} \quad T_b\psi(x) = -i\psi_x(x).$$

A minimizing state is then given by

$$\psi(x) = (1-\rho x)^{i\mu_a - \frac{1}{2} - i\frac{\mu_b}{\rho} - i\frac{\kappa}{\rho}} e^{-i\kappa x}. \tag{21}$$

Suppose that $\rho = ir,\ \ r \in \mathbb{R}$*. Then, the minimizing state is square integrable if* $r < 0$ *and* $\mu_b < -\kappa$ *or if* $r > 0$ *and* $\mu_b > -\kappa$*.*

Proof: The proof can be performed by following the lines of the proof of Lemma 2. This time, the corresponding differential equation is given by

$$\psi_x(x) = \frac{\left(\mu_b - \frac{i\rho}{2} + \rho\kappa x - \rho\mu_a\right)}{i(\rho x - 1)}\psi, \tag{22}$$

where ρ is purely imaginary. Eq. (22) can again be solved by separation of variables which leads to (21). $\square$

In Figure 2 we have plotted the minimizing ψ, for this choice of a section for the AWH group in $1D$.
At $\kappa = 1$ the current solution reduces to

$$\psi(x) = (1-\rho x)^{i\mu_a - \frac{1}{2} - i\frac{\mu_b}{\rho} - i\frac{1}{\rho}} e^{-ix}. \tag{23}$$

It is interesting to compare this solution to the one obtained for the previously used section $a = \beta(\omega)$ for $\alpha = 1$, which yields:

$$\psi(x) = (1+\lambda x)^{i\frac{\mu_b}{\lambda} - \frac{1}{2} - i\mu_\omega + \frac{i}{\lambda}} e^{-ix} \quad . \tag{24}$$

The constraints for the two solutions to agree for $\alpha = 1$ are derived as follows

$$\lambda = -\rho, \quad \mu_a = -\mu_\omega \quad .$$

4 The $2D$ Affine Weyl-Heisenberg Group

In this section we are interested in finding uncertainty minimizers for the two-dimensional affine Weyl-Heisenberg group with a generating element $(A, \omega, b, \phi),\ \ \omega, b \in \mathbb{R}^2, A \in Gl(2, \mathbb{R}), \phi \in \mathbb{R}$ and group law

$$(b, \omega, A, \phi) \circ (b', \omega', A', \phi') = (b + Ab', \omega + A^{-1}\omega', AA', \phi + \phi' + \omega^T Ab').$$

The Stone - von Neumann representation of the AWH group in two dimensions is given by:

$$[U(b, \omega, A, \phi)\psi](x, y) = \frac{1}{|\det(A)|} e^{i(\omega_x(x-b_x) + \omega_y(y-b_y) + \phi)} \psi\left(A(x - b_x, y - b_y)\right) \tag{25}$$

with the unimodular Haar measure

$$\frac{1}{|\det(A)|^2} db d\omega dm(A) d\phi,$$

where $dm(A)$ denotes a usual measure when parametrizing the matrix A. In our discussion we explore various subgroups of the full $2D$ AWH group, starting from the $SIM(2)$ group, and moving to the full group structure.

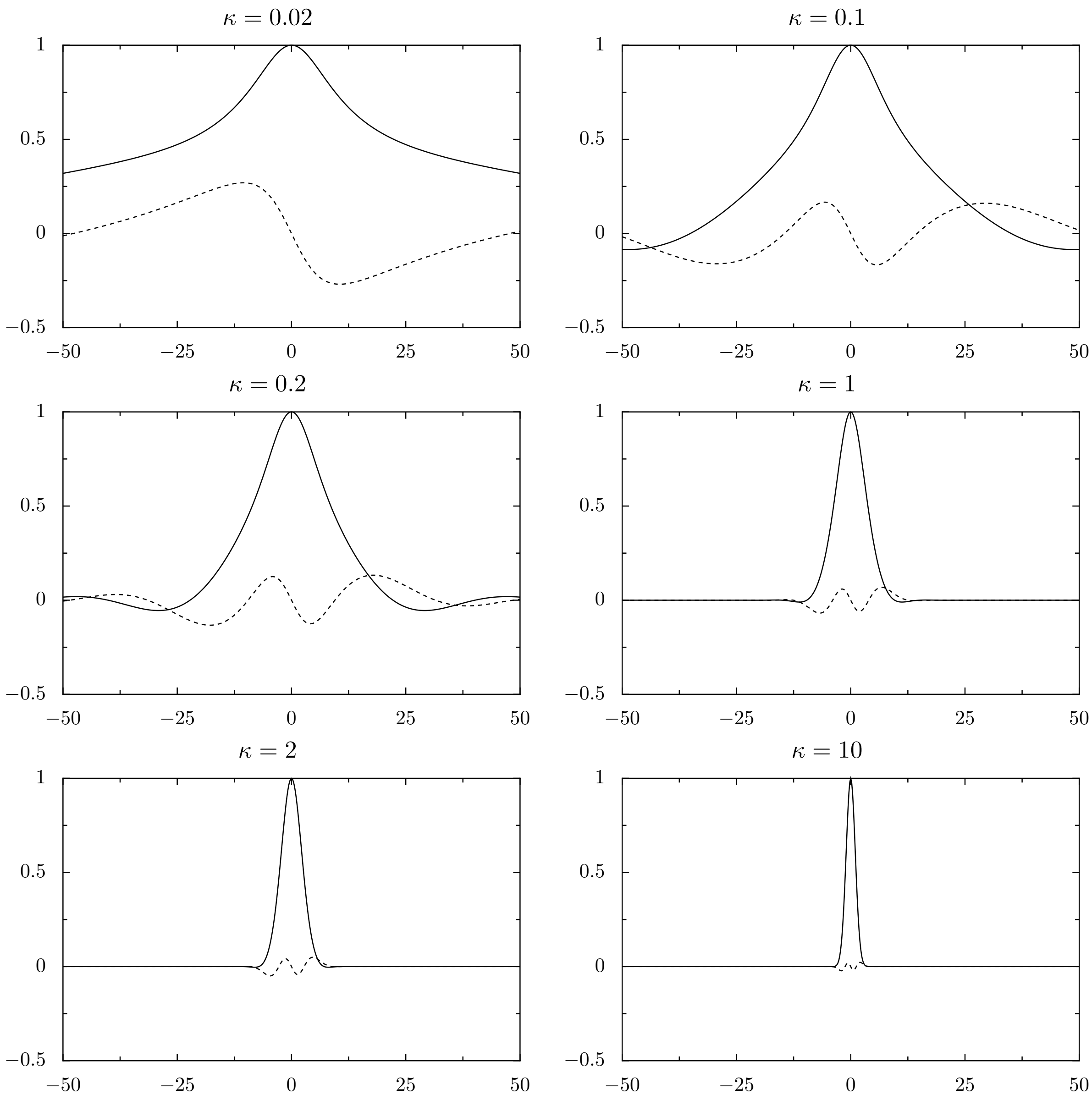

Figure 2: The minimizers of the AWH uncertainty for $\rho = 0.1i$, $\mu_a = \mu_b = 0$ and different values of κ. The real part is plotted solid, the imaginary part is dashed. Note the transition from a Cauchy wavelet for small κ (affine case) to a Gaussian for large κ (Weyl-Heisenberg case).

4.1 The $2D$ Similitude Weyl-Heisenberg Subgroup

We start our discussion by considering the similitude group, which only allows rotations and scalings $A = A(a, \theta)$. The general representation of the $2D$ similitude Weyl-Heisenberg subgroup is given by

$$[U(b, \omega, (a, \theta), 0)\psi](x, y) = \frac{1}{a} e^{i(\omega_x(x-b_x)+\omega_y(y-b_y))} \psi \left(\tau_\theta \left(\frac{x - b_x}{a}, \frac{y - b_y}{a} \right) \right), \tag{26}$$

where $\tau_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. This representation fails to be square integrable [15], therefore, we are faced with the interesting question of selecting an appropriate section. To this end, let $H = (0, 0, (a, 0), \phi)$, consider $G_{AWH} \setminus H$ and take the section

$$\sigma(b, \omega, \theta) = (b, \omega, (\Phi(\omega), \theta), 0).$$

We consider a coupling between the frequency ω and the scaling a by $a = \Phi(\omega)$. More specifically, in the spirit of the α-modulation spaces framework, we assume that the function Φ depends only on the p-norm of the frequency vector ω

$$\Phi(\omega) = \frac{1}{(1 + \|\omega\|_p)^\alpha}. \tag{27}$$

As before, we would like to obtain the infinitesimal generators of this group by calculating the appropriate derivatives of the representation of this group at the identity element. Depending on the choice of the section $\Phi(\omega)$, we will obtain different infinitesimal operators from the partial derivatives with respect to ω_k:

$$\tilde{T}_{\omega_k} = (T_{\omega_k} + \Phi'_{\omega_k}(0) T_a), \tag{28}$$

for $k \in \{x, y\}$. Therefore, we have to estimate the derivative of Φ at $\omega = 0$. Our particular choice of Φ yields

$$\begin{aligned} \Phi'_{\omega_k}(\omega) &= -\alpha(1 + \|\omega\|_p)^{-\alpha-1} \frac{1}{p} \|\omega\|_p^{1-p} p \omega_k^{p-1} \operatorname{sign}(\omega_k) \\ &= \frac{-\alpha}{(1 + \|\omega\|_p)^{\alpha+1}} \left[\frac{\omega_k^p}{\sum_j |\omega_j|^p} \right]^{\frac{p-1}{p}} \operatorname{sign}(\omega_k). \end{aligned} \tag{29}$$

Next, we have to evaluate this expression at $\omega_k = 0$ for all k. In contrast to the $1D$ situation, the resulting infinitesimal operators and the corresponding commutation relations strongly depend on the choice of p. We start by selecting the L_1-norm, as this allows a straight forward calculation of infinitesimal generators.

4.1.1 AWH Minimizers Using the L_1-Norm

In this case: $a = \Phi(\omega) = \frac{1}{(1+|\omega_x|+|\omega_y|)^\alpha}$, thus the representation becomes:

$$[U(b, \omega, \Phi(\omega), \theta, 0)\psi](x, y) = (1+|\omega_x|+|\omega_y|)^\alpha e^{i((x-b_x)\omega_x+(y-b_y)\omega_y)} \psi \left((1 + |\omega_x| + |\omega_y|)^\alpha \tau_\theta(x - b_x, y - b_y) \right). \tag{30}$$

The self-adjoint infinitesimal operators are given by:

$$\begin{aligned} T_{\omega_x}\psi(x, y) &= (i\alpha - x)\psi(x, y) + i\alpha(x\psi_x(x, y) + y\psi_y(x, y)), \\ T_{\omega_y}\psi(x, y) &= (i\alpha - y)\psi(x, y) + i\alpha(x\psi_x(x, y) + y\psi_y(x, y)), \\ T_{b_x}\psi(x, y) &= -i\psi_x(x, y), \\ T_{b_y}\psi(x, y) &= -i\psi_y(x, y), \\ T_\theta\psi(x, y) &= i(y\psi_x(x, y) - x\psi_y(x, y)). \end{aligned} \tag{31}$$

Out of the ten commutation relations, three vanish, $[T_{\omega_x}, T_{b_y}] = 0$, $[T_{\omega_y}, T_{b_x}] = 0$, $[T_{b_x}, T_{b_y}] = 0$, and we are left with seven partial differential equations.

1.

$$(i\alpha-x)\psi(x, y)+i\alpha(x\psi_x(x, y)+y\psi_y(x, y))-\mu_{\omega_x}\psi(x, y) = \lambda_1 \left((i\alpha - y)\psi(x, y) + i\alpha(x\psi_x(x, y) + y\psi_y(x, y)) - \mu_{\omega_y}\psi(x, y) \right)$$

2.

$$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi(x,y) = \lambda_2\left(-i\psi_x(x,y) - \mu_{b_x}\psi(x,y)\right)$$

3.

$$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi(x,y) = \lambda_3\left(i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi(x,y)\right)$$

4.

$$(i\alpha - y)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_y}\psi(x,y) = \lambda_4\left(-i\psi_y(x,y) - \mu_{b_y}\psi(x,y)\right)$$

5.

$$(i\alpha - y)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_y}\psi(x,y) = \lambda_5\left(i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi(x,y)\right)$$

6.

$$i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi(x,y) = \lambda_6\left(-i\psi_x(x,y) - \mu_{b_x}\psi(x,y)\right)$$

7.

$$i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi(x,y) = \lambda_7\left(-i\psi_y(x,y) - \mu_{b_y}\psi(x,y)\right)$$

The only simultaneous solution to these equations is the trivial one, $\psi = 0$ everywhere. Therefore, we aim at finding a partial solution to this set of equations, which involves operators from the enveloping algebra. First of all, we impose rotational invariance.

Suppose that the minimizer is of the form $g(r)$ where $r = \sqrt{x^2 + y^2}$. Then, we consider the following infinitesimal operators with respect to $g(r)$:

$$\begin{aligned} T_\theta g(r) &= 0, \\ T_b g(r) &= (T_{b_x}^2 + T_{b_y}^2)g(r) = -\frac{d^2 g}{dr^2}(r) - \frac{1}{r}\frac{dg}{dr}(r). \end{aligned}$$

Moreover, the operators $T_{\omega_x}, T_{\omega_y}$ are commuting with respect to $g(r)$,i.e., $[T_{\omega_x}, T_{\omega_y}]g(r) = 0$. These observations lead to two possible solutions: the first involves defining a new operator: $T_\omega = T_{\omega_x}T_{\omega_y} - T_{\omega_y}T_{\omega_x}$ and considering its commutator relations with T_θ and T_b. Then, any function $g(r)$ that is rotation invariant is a valid minimizer of the uncertainties related to these operators.

Another option is to consider T_{ω_x} and T_{ω_y} with respect to $g(r)$. The commutators of these operators with T_b are not equal to zero and we obtain the differential equation

$$g_{rr}(r) + \frac{1}{r}g_r(r) + \mu_b g(r) = 0 \tag{32}$$

whose solution is given by Bessel functions of the first and second kind:

$$\psi(r) = c_1 J_0(\sqrt{\mu_b}r) + c_2 Y_0(\sqrt{\mu_b}r). \tag{33}$$

Nevertheless, this solution is not square integrable.

Another interesting effort is to find a solution for a single differential equation and thus obtain a selective minimal uncertainty with respect to two operators only. For example, let us consider equation 2 only:

$$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi(x,y) = \lambda_2\left(-i\psi_x(x,y) - \mu_{b_x}\psi(x,y)\right) \quad .$$

Choosing $\lambda_2 = 0$ yields

$$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi(x,y) = 0.$$

A possible solution is given by the expression:

$$\psi(x,y) = y^{-i\frac{\mu_{\omega_x}}{\alpha} - 1} e^{-i\frac{x}{\alpha}} \tau\left(\frac{x}{y}\right), \tag{34}$$

where τ is an arbitrary function of the variable $\frac{x}{y}$. A particular solution is depicted in Figure 3.

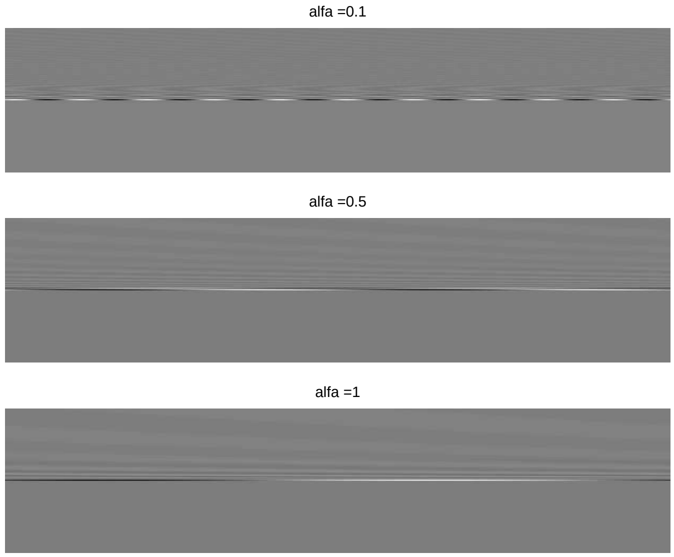

Figure 3: A possible minimizing function for $\tau\left(\frac{x}{y}\right)=1$.

4.1.2 AWH Minimizers Using the L_2-Norm

As we have seen in the previous section, the choice $a=\Phi(\omega)=\left(1+\sqrt{\omega_x^2+\omega_y^2}\right)^{-\alpha}$ proves to be futile. It is interesting to explore another solution to the problem mentioned in Section 4.1, where the relationship between the scale and frequency is given by

$$a=\Phi(\omega)=\left(1+\omega_x^2+\omega_y^2\right)^{-\alpha} \quad .$$

The unitary representation induced by this section of the similitude Weyl-Heisenberg group is then given by

$$[U(b,\omega,(\Phi(\omega),\theta),0)\psi](x,y)=(1+\omega_x^2+\omega_y^2)^{\alpha}e^{i((x-b_x)\omega_x+(y-b_y)\omega_y)}\psi\left((1+\omega_x^2+\omega_y^2)^{\alpha}\tau_\theta(x-b_x,y-b_y)\right), \quad (35)$$

and τ_θ is the same as already defined. The infinitesimal generators are then given by:

$$\begin{aligned}
T_{\omega_x}\psi(x,y) &= -x\psi(x,y),\\
T_{\omega_y}\psi(x,y) &= -y\psi(x,y),\\
T_{b_x}\psi(x,y) &= -i\psi_x(x,y),\\
T_{b_y}\psi(x,y) &= -i\psi_y(x,y),\\
T_\theta\psi(x,y) &= i(y\psi_x(x,y)-x\psi_y(x,y)). \qquad (36)
\end{aligned}$$

It is interesting to note that the dependency on the parameter α has disappeared. This means that selecting this type of section may provide a solution regardless of the smoothness space we are dealing with.

The differential equations resulting from the non-commuting operators are:

$$\begin{aligned}
-x\psi(x,y)-\mu_{\omega_x}\psi(x,y) &= \lambda_1(-i\psi_x(x,y)-\mu_{b_x}\psi(x,y)),\\
-x\psi(x,y)-\mu_{\omega_x}\psi(x,y) &= \lambda_2(i(y\psi_x(x,y)-x\psi_y(x,y))-\mu_\theta\psi(x,y)),\\
-y\psi(x,y)-\mu_{\omega_y}\psi(x,y) &= \lambda_3(-i\psi_y(x,y)-\mu_{b_y}\psi(x,y)),\\
-y\psi(x,y)-\mu_{\omega_y}\psi(x,y) &= \lambda_4(i(y\psi_x(x,y)-x\psi_y(x,y))-\mu_\theta\psi(x,y)),\\
-i\psi_x(x,y)-\mu_{b_x}\psi(x,y) &= \lambda_5(i(y\psi_x(x,y)-x\psi_y(x,y))-\mu_\theta\psi(x,y)),\\
-i\psi_y(x,y)-\mu_{b_y}\psi(x,y) &= \lambda_6(i(y\psi_x(x,y)-x\psi_y(x,y))-\mu_\theta\psi(x,y)). \qquad (37)
\end{aligned}$$

If, again, we search for a solution which is rotational invariant, i.e. $\psi(x,y)=g(r)$, we may satisfy all equations that involve the operator T_θ. Moreover, applying restrictions to our parameters, e.g. $\lambda_1=\lambda_3=$

$\lambda, \mu_{b_x} = \mu_{b_y}, \omega_x = \omega_y$, we may obtain a rotationally invariant solution to the first and third equation as well, which is a Gaussian function

$$\psi = e^{-\frac{i}{\lambda}\left(\frac{x^2+y^2}{2}\right)}. \tag{38}$$

4.2 AWH Minimizers with Anisotropic Scaling

In the previous treatment of the two-dimensional case, we regard the frequency as a vector, but treat the scale as a scalar argument. Moreover, we use the $SIM(2)$ group rather than the full affine group. We are interested to add more degrees of freedom to our setting, and as a first step we observe a relationship where the scale is also a two dimensional vector and not a scalar

$$a_x = \beta_1(\omega_x), \qquad a_y = \beta_2(\omega_y).$$

We explore a generalization to two dimensions of the one-dimensional affine Weyl-Heisenberg group, and ignore for now rotation and shear. We thus consider the group with a generic element $g = (b_x, b_y, \omega_x, \omega_y, a_x, a_y, \phi,)$ where $b_x, b_y, \omega_x, \omega_y, \phi \in \mathbb{R}$ and $a_x, a_y \in \mathbb{R}_+$ equipped with a group law

$$(b_x, b_y, \omega_x, \omega_y, a_x, a_y, \phi) \circ (b'_x, b'_y, \omega'_x, \omega'_y, a'_x, a'_y, \phi') =$$

$$(b_x + a_x b'_x, b_y + a_y b'_y, \omega_x + a_x^{-1}\omega'_x, \omega_y + a_y^{-1}\omega'_y, a_x a'_x, a_y a'_y, \phi + \phi' + \omega_x a_x b'_x + \omega_y a_y b'_y).$$

This is a subgroup of the $2D$ AWH group. The inverse element of $g \in G$ is given by

$$g^{-1} = (-a_x^{-1} b_x, -a_y^{-1} b_y, -a_x\omega_x, -a_y\omega_y, a_x^{-1}, a_y^{-1}, -\phi + b_x\omega_x + b_y\omega_y). \tag{39}$$

Let us look at the following representation

$$[U(b_x, b_y, \omega_x, \omega_y, a_x, a_y, \phi)\psi](x,y) = \frac{1}{\sqrt{a_x a_y}} e^{i((x-b_x)\omega_x + (y-b_y)\omega_y)+\phi}\psi\left(\frac{x-b_x}{a_x}, \frac{y-b_y}{a_y}\right) \tag{40}$$

which is the $2D$ extension of the Stone-von-Neumann representation of the $1D$ AWH group. This representation fails to be square integrable, and therefore we restrict ourselves to the homogeneous space $G_{AWH}\backslash H$ with

$$H := (0,0,0,0,a_x,a_y,\phi) \in G_{AWH}. \tag{41}$$

Next, we consider the section $\sigma(b_x, b_y, \omega_x, \omega_y) = (b_x, b_y, \omega_x, \omega_y, \beta_x(\omega_x), \beta_y(\omega_y), 0)$, and would like to prove that this section is admissible.

We define a self-adjoint operator $A_\sigma f$

$$\begin{aligned}(A_\sigma f)(x,y) &:= \int_X \langle f, U(\sigma(h))\psi\rangle U(\sigma(h))\psi d\mu(h) \\ &= \int\int\int\int \langle f, \psi_{\omega_x,\omega_y,\beta_x(\omega_x),\beta_y(\omega_y),b_x,b_y}\rangle \psi_{\omega_x,\omega_y,\beta_x(\omega_x),\beta_y(\omega_y),b_x,b_y}(x,y) db_x db_y d\omega_x d\omega_y. \end{aligned} \tag{42}$$

It can be written as a Fourier multiplier operator

$$\widehat{(A_\sigma f)} = m_{\beta_x,\beta_y}\hat{f} \tag{43}$$

where

$$m_{\beta_x,\beta_y}(\gamma_x,\gamma_y) = \int\int |\hat{\psi}(\beta_x(\omega_x)(\gamma_x - \omega_x), \beta_y(\omega_y)(\gamma_y - \omega_y))|^2 \beta_x(\omega_x)\beta_y(\omega_y) d\omega_x d\omega_y. \tag{44}$$

Next, we follow the lines of [4] to show that m_{β_x,β_y} is bounded from above and below, i.e.

$$C_1 \leq m_{\beta_x,\beta_y} \leq C_2 \tag{45}$$

for constants $0 < C_1 < C_2 < \infty$. We start with the following lemma which is a straight forward generalization of Lemma 5.1 in [4] to the $2D$-case. Therefore we omit the details.

Lemma 5 *Consider the specific section σ given by the functions*

$$\beta_x(\omega_x) = \beta_{x,\alpha_x}(\omega_x) = (1+|\omega_x|)^{-\alpha_x},$$
$$\beta_y(\omega_x) = \beta_{y,\alpha_y}(\omega_y) = (1+|\omega_y|)^{-\alpha_y}.$$

Let us define

$$r_{\gamma_x}(\omega_x) := \beta_x(\omega_x)(\gamma_x - \omega_x) = (1+|\omega_x|)^{-\alpha_x}(\gamma_x - \omega_x),$$
$$r_{\gamma_y}(\omega_y) := \beta_y(\omega_y)(\gamma_y - \omega_y) = (1+|\omega_y|)^{-\alpha_y}(\gamma_y - \omega_y).$$

Then, for any fixed $A > 0$, there exist $\gamma_{x,A}, \gamma_{y,A} > 0$ such that for all $\gamma_x \geq \gamma_{x,A}, \gamma_y \geq \gamma_{y,A}$ the functions $r_{\gamma_x}, r_{\gamma_y}$ are invertible on

$$A_{\omega_x} = \{\omega_x : r_{\gamma_x}(\omega_x) \in [-A, A]\} \quad \textit{and} \quad A_{\gamma_y} = \{\omega_y : r_{\gamma_y}(\omega_y) \in [-A, A]\}$$

respectively. The inverse functions $r_{\gamma_x}^{-1}, r_{\gamma_y}^{-1}$ of $r_{\gamma_x}, r_{\gamma_y}$ on $[-A, A]$ have the form

$$r_{\gamma_x}^{-1} = -xg_1(\gamma_x, x) + \gamma_x, \qquad r_{\gamma_y}^{-1} = -yg_2(\gamma_y, y) + \gamma_y,$$

with some functions $g_1(\gamma_x, x), g_2(\gamma_y, y)$ satisfying

$$xg_1(\gamma_x, x) + g_1(\gamma_x, x)^{\frac{1}{\alpha_x}} = 1 + \gamma_x \quad \textit{and} \quad yg_2(\gamma_y, y) + g_2(\gamma_y, y)^{\frac{1}{\alpha_y}} = 1 + \gamma_y.$$

Furthermore, g_1, g_2 fulfill

$$\lim_{\gamma_x \to \infty} \gamma_x^{-\alpha_x} g_1(\gamma_x, x) = 1, \qquad \lim_{\gamma_y \to \infty} \gamma_y^{-\alpha_y} g_2(\gamma_y, y) = 1$$

uniformly for $x, y \in [-A, A]$.

Theorem 6 *Let the Borel section σ be given by $\sigma(b_x, b_y, \omega_x, \omega_y) = (b_x, b_y, \omega_x, \omega_y, \beta_x(\omega_x), \beta_y(\omega_y), 0)$ with $\beta_x(\omega_x) = (1+|\omega_x|)^{-\alpha_x}, \beta_y(\omega_y) = (1+|\omega_y|)^{-\alpha_y}$. Let ψ be a non zero L_2 function whose Fourier transform is compactly supported. Then, ψ is admissible, i.e., the condition*

$$C_1 \leq m_{\beta_x,\beta_y}(\gamma_x, \gamma_y) \leq C_2$$

is satisfied for $0 < C_1 \leq C_2 < \infty$.

Proof. The proof can be performed by following the lines of the proof of Theorem 5.2 in [4]. For reader's convenience, we briefly sketch the arguments. We consider the case where either γ_x or γ_y tend to $+\infty$. Let us assume that $\text{supp}(\hat{\psi}) \subset [-A, A] \times [-A, A]$. We substitute $x = r_{\gamma_x}(\omega_x), y = r_{\gamma_y}(\omega_y)$ for $\gamma_x \geq \gamma_{x,A} > 0, \gamma_y \geq \gamma_{y,A} > 0$ in the expression for $m_{\beta_x,\beta_y}(\gamma_x, \gamma_y)$ to obtain

$$\begin{aligned} m_{\beta_x,\beta_y}(\gamma_x, \gamma_y) &= \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{\psi}(r_{\gamma_x}(\omega_x), r_{\gamma_y}(\omega_y))|^2 \beta_x(\omega_x)\beta_y(\omega_y) d\omega_x d\omega_y \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{\psi}(x, y)|^2 \beta_x(r_{\gamma_x}^{-1}(x))\beta_y(r_{\gamma_y}^{-1}(y))(r_{\gamma_x}^{-1})'(r_{\gamma_y}^{-1})' dx dy. \end{aligned} \tag{46}$$

Next, we calculate the values of the derivatives of the inverse functions $r_{\gamma_x}^{-1}, r_{\gamma_y}^{-1}$ using

$$r'_{\gamma_x}(\omega_x) = \beta'_x(\omega_x)(\gamma_x - \omega_x) - \beta_x(\omega_x) = -\beta_x(\omega_x)\left(\alpha_x \frac{\gamma_x - \omega_x}{1 + \omega_x} + 1\right),$$
$$r'_{\gamma_y}(\omega_y) = \beta'_y(\omega_y)(\gamma_y - \omega_y) - \beta_y(\omega_y) = -\beta_y(\omega_y)\left(\alpha_y \frac{\gamma_y - \omega_y}{1 + \omega_y} + 1\right)$$

to obtain

$$(r_{\gamma_x}^{-1})'(x) = \frac{1}{r'_{\gamma_x}(r_{\gamma_x}^{-1}(x))} = -\frac{1}{\beta_x(r_{\gamma_x}^{-1}(x)\left(1 + \alpha_x \frac{\gamma_x - r_{\gamma_x}^{-1}(x)}{1 + r_{\gamma_x}^{-1}(x)}\right)},$$
$$(r_{\gamma_y}^{-1})'(y) = \frac{1}{r'_{\gamma_y}(r_{\gamma_y}^{-1}(y))} = -\frac{1}{\beta_y(r_{\gamma_y}^{-1}(y)\left(1 + \alpha_y \frac{\gamma_y - r_{\gamma_y}^{-1}(y)}{1 + r_{\gamma_y}^{-1}(y)}\right)}.$$

Thus, for values of $\gamma_x \geq \gamma_{x,A} > 0, \gamma_y \geq \gamma_{y,A} > 0$ we have

$$m_{\beta_x,\beta_y}(\gamma_x,\gamma_y) = \int_{-A}^{A}\int_{-A}^{A} |\hat{\psi}(x,y)|^2 G(\gamma_x,\gamma_y,x,y)dxdy \tag{47}$$

where

$$G(\gamma_x,\gamma_y,x,y) = \frac{1}{\left(1+\alpha_x \frac{\gamma_x - r_{\gamma_x}^{-1}(x)}{1+r_{\gamma_x}^{-1}(x)}\right)} \frac{1}{\left(1+\alpha_y \frac{\gamma_y - r_{\gamma_y}^{-1}(y)}{1+r_{\gamma_y}^{-1}(y)}\right)} = \frac{1}{1+\alpha_x x g_1(\gamma_x,x)^{1-\frac{1}{\alpha_x}}} \frac{1}{1+\alpha_y y g_2(\gamma_y,y)^{1-\frac{1}{\alpha_y}}},$$

where we have used the definitions in the previous lemma. According to this lemma, we may substitute $\gamma_x^{\alpha_x}$ for $g_1(\gamma_x,x)$ when γ_x goes to infinity, and the same for $g_2(\gamma_y,y)$ when $\gamma_y \to \infty$

$$\begin{aligned}
\lim_{\gamma_x\to\infty,\gamma_y\to\infty} G(\gamma_x,\gamma_y,x,y) &= \lim_{\gamma_x\to\infty,\gamma_y\to\infty} \frac{1}{1+\alpha_x x g_1(\gamma_x,x)^{1-\frac{1}{\alpha_x}}} \frac{1}{1+\alpha_y y g_2(\gamma_y,y)^{1-\frac{1}{\alpha_y}}} \\
&= \lim_{\gamma_x\to\infty,\gamma_y\to\infty} \frac{1}{1+\alpha_x x \gamma_x^{\alpha_x(1-\frac{1}{\alpha_x})}} \frac{1}{1+\alpha_y y \gamma_y^{\alpha_y(1-\frac{1}{\alpha_y})}} \\
&= 1,
\end{aligned}$$

and therefore we finally have

$$\lim_{\gamma_x\to\infty,\gamma_y\to\infty} m_{\beta_x,\beta_y}(\gamma_x,\gamma_y) = \int_{-A}^{A}\int_{-A}^{A} |\hat{\psi}(x,y)|^2 dxdy \tag{48}$$

for any L_2-function with compact support in the Fourier domain, and thus we obtain that m_{β_x,β_y} is bounded from below and above. $\square$

Now, that this section is proven to be admissible, we would like to explore the uncertainty principle minimizers associated with this representation. We assume that it should be a two-dimensional extension of the one-dimensional solution obtained earlier. The representation for the quotient as a function of α_x, α_y is then given by:

$$\begin{gathered}
[U(b_x,b_y,\omega_x,\omega_y,\beta_x(\omega_x),\beta_y(\omega_y),0)\psi](x,y) = \\
(1+|\omega_x|)^{\frac{\alpha_x}{2}}(1+|\omega_y|)^{\frac{\alpha_y}{2}} e^{i(\omega_x(x-b_x)+\omega_y(y-b_y))}\psi\left((1+|\omega_x|)^{\alpha_x}(x-b_x),(1+|\omega_y|)^{\alpha_y}(y-b_y)\right).
\end{gathered}$$

From this representation we may see that the x and y axes are not correlated, and thus we obtain the following infinitesimal generators

$$\begin{aligned}
(T_{b_x}\psi)(x,y) &= -\frac{\partial}{\partial x}\psi(x,y), \\
(T_{b_y}\psi)(x,y) &= -\frac{\partial}{\partial y}\psi(x,y), \\
(T_{\omega_x}\psi)(x,y) &= (\frac{\alpha_x}{2}+i)\psi(x,y) + \alpha_x x\frac{\partial}{\partial x}\psi(x,y), \\
(T_{\omega_y}\psi)(x,y) &= (\frac{\alpha_y}{2}+i)\psi(x,y) + \alpha_y y\frac{\partial}{\partial y}\psi(x,y).
\end{aligned} \tag{49}$$

In order to make these operators self-adjoint, we multiply them by i. The commutators between the x and y operators vanish, and we have to solve two independent one-dimensional problems, with the following solutions

$$\psi(x,y) = (\alpha_x\lambda_x x+1)^{-\frac{1}{2}-\frac{i\mu_{\omega_x}}{\alpha_x}+\frac{i\mu_{b_x}}{\alpha_x\lambda_x}+\frac{i}{\alpha_x^2\lambda_x}} e^{\frac{-ix}{\alpha_x}} (\alpha_y\lambda_y y+1)^{-\frac{1}{2}-\frac{i\mu_{\omega_y}}{\alpha_y}+\frac{i\mu_{b_y}}{\alpha_y\lambda_y}+\frac{i}{\alpha_y^2\lambda_y}} e^{\frac{-iy}{\alpha_y}}. \tag{50}$$

In order for this solution to be square integrable, the following should be met: we denote $\lambda_x = i\varpi_x, \lambda_y = i\varpi_y$ where $\varpi_x, \varpi_y \in \mathbb{R}$. Then, if $\varpi_x, \varpi_y < 0$, then $\mu_{b_x} > -\frac{1}{\alpha_x}, \mu_{b_y} > -\frac{1}{\alpha_y}$. If $\varpi_x, \varpi_y > 0$, then $\mu_{b_x} < -\frac{1}{\alpha_x}, \mu_{b_y} < -\frac{1}{\alpha_y}$.

5 Discussion and Conclusions

The STFT and wavelet transform can both be viewed as the integral transforms related to the Weyl-Heisenberg and affine group, respectively. From a signal processing viewpoint, it is interesting to ask whether there is a combined integral transform that smoothly interpolates between these two. In a recent publication [4] an answer to this question was given from a harmonic analysis viewpoint. This work deals with coorbit spaces that are associated with some group. These spaces contain all functions for which the associated integral transform is contained in some weighted L_p-space. The coorbit spaces associated with the Weyl-Heisenberg group are the modulation spaces [7] and those associated with the affine group are the Besov spaces. In [4] it was proposed to construct some mixed smoothness spaces that lie in between the Besov and modulation spaces and to use the affine Weyl-Heisenberg group as it contains both groups. Since this group is known to have no representation which is square integrable, it was offered to factor out a suitable closed subgroup and work with quotients.

We adapted this setting, and analyzed in this work several possible sections in the framework of the affine Weyl-Heisenberg setting. We explored the uncertainty relations associated with various selections of sections, and aimed at providing their minimizers.

The sections we have explored in this manuscript are those that provide inverse relations between the scale and frequency attributes. This is a natural framework, as we expect high frequency phenomena to manifest themselves within a small scale and vise-versa. Moreover, using the α-modulation spaces, we can smoothly move from the Gabor (Weyl-Heisenberg) transform, via the Gabor wavelet transform to the wavelet (affine) transform. In the one-dimensional case, we have considered the section which regards the scale as a function of frequency. This section is known to be admissible, and we have explicitly calculated the minimizer with respect to both time and frequency localization. However, using this section, we can only treat the Gabor and Gabor wavelet transform. Therefore, the section in which the frequency is regarded as a function of the scale was considered. It turns out that this section is not admissible. Still, as it can be treated in the framework of quasi-coherent states, we have also calculated the minimizers for this case. We have also found that for the Gabor-wavelet case, the two sections may provide the same solution under some constraints.

In the two-dimensional case there are several possible sections that can be selected. We have presented in this manuscript the similitude Weyl-Heisenberg group and treated it with respect to the L_1-norm and the squared L_2-norm. We also regarded the case where the scale is also considered as a vector. As it is not possible to find a minimizer for all the uncertainty relations, we may consider a single such relation (e.g. the minimizer with respect to the frequency in the x direction and rotation), or we could find a rotation invariant solution when we consider elements of the enveloping algebra.

To summarize, we have considered possible uncertainty minimizers for the AWH group, and have especially regarded the attribute of interpolating between the Gabor and wavelet transforms. The applicative significance of these minimizers is still not well established. However, based on the importance that the Gabor functions have in signal and image processing, we believe that this significance awaits discovering.

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