# A Tikhonov–Based Projection Iteration for Nonlinear Ill–Posed Problems with Sparsity Constraints

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**Abstract**

In this paper, we consider nonlinear inverse problems where the solution is assumed to have a sparse expansion with respect to a preassigned basis or frame. We develop a scheme which allows to minimize a Tikhonov functional where the usual quadratic regularization term is replaced by a one–homogeneous (typically weighted $\ell_p$) penalty on the coefficients (or isometrically transformed coefficients) of such expansions. For $p < 2$, the regularized solution will have a sparser expansion with respect to the basis or frame under consideration. The computation of the regularized solution amounts in our setting to a Landweber–fixed–point iteration with a projection applied in each fixed–point iteration step. The performance of the resulting numerical scheme is demonstrated by solving the nonlinear inverse SPECT (Single Photon Emission Computerized Tomography) problem.

## 1 Introduction – the scope of the problem

We consider the computation of an approximation to a solution of a nonlinear operator equation

$$T(x) = y \ , \tag{1.1}$$

where $T : X \to Y$ is an ill-posed operator between Hilbert spaces $X, Y$. If only noisy data $y^\delta$ with

$$\|y^\delta - y\| \leq \delta \tag{1.2}$$

are available, problem (1.1) has to be stabilized by regularization methods. In recent years, many of the well known methods for linear ill-posed problems have been generalized to nonlinear operator equations. But so far all the proposed schemes for nonlinear problems incorporate at most quadratic regularization. In many applications the solution is assumed to have sparse expansion with respect to some preselected frame (or basis). This immediately leads to the

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involvement of non–quadratic penalties, e.g. $\ell_p$ norms with $p < 2$. In linear lore, this problem is already solved, see [7]. In this paper, we aim now to carry over the theory to nonlinear inverse problems and extend it to more general sparsity constraints. More general sparsity constraints mean here being no longer restricted to weighted $\ell_p$ norms as considered in [7]. We consider here the wide range of one–homogeneous and convex constraints, where the $\ell_1$ norm is just one famous example for a sparsity constraint. Another famous one–homogeneous constraint is the $TV$ semi norm which is very often used in image processing when aiming to reconstruct sharp boundaries and edges in the given image, see e.g. [1, 13, 14, 15, 22]. Since we focus here on constraints on the basis or frame coefficients of the function to be reconstructed, $TV$–like constraints are not directly applicable in our context. However, there is a remarkable relation between $TV$ penalties and one–homogeneous constraints on the frame or wavelet basis coefficients which can be explained by the inclusion $B^1_{1,1} \subset BV \subset B^1_{1,1} - weak$ (in two dimensions), see for further Harmonic analysis on $BV$ [4, 5]. This relation yields a wavelet/frame–based near $BV$ reconstruction when limiting to Haar frames and using a $B^1_{1,1}$ constraint, see for further elaboration [8, 9]. But we want to be not too restrictive and allow thus to chose general one–homogeneous and convex functionals on the frame coefficients or its isometrically transformed versions.

Assume now we are given some preassigned frame $\{\phi_\lambda\}_{\lambda\in\Lambda} \subset X$ for which we have some associated frame operator $F : X \to \ell_2$ via $Fx = \{\langle x, \phi_\lambda\rangle\}_{\lambda\in\Lambda}$ with $A \cdot I \leq F^*F \leq B \cdot I$. Assuming, moreover, a Gaussian error noise model for the data misfit term, the variational formulation of the nonlinear inverse problem with sparsity, or more general, one–homogeneous constraints can be casted as follows: find a sequence of coefficients $\boldsymbol{g} \in \ell_2$ such that

$$J_\alpha(\boldsymbol{g}) = \|y^\delta - T(F^*\boldsymbol{g})\|^2_Y + 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}) \tag{1.3}$$

is minimized. Here $\Psi$ stands for some positive, one–homogeneous, lower semi–continuous and convex penalty (which is usually some weighted $\ell_p$ norm of the frame coefficients), and the infinite matrix $\boldsymbol{L}$ is restricted to be an isometric mapping. In particular, we also need to require,

$$\|\boldsymbol{g}\|_{\ell_2} \leq \Psi(\boldsymbol{L}\boldsymbol{g}). \tag{1.4}$$

The strategies for nonlinear cases where quadratic penalties are well suited, suggested in [21], seem to be also adequate when dealing with sparsity, or more general, with one–homogeneous constraints. The idea goes as follows: we replace (1.3) by a sequence of functionals from which we hope that they are easier to treat and that the sequence of minimizers converge in some sense to, at least, a critical point of (1.3). To be more concrete, for some auxiliary $\boldsymbol{a} \in \ell_2$, we introduce the following surrogate functional

$$J^s_\alpha(\boldsymbol{g}, \boldsymbol{a}) := J_\alpha(\boldsymbol{g}) + C\|\boldsymbol{g} - \boldsymbol{a}\|^2_{\ell_2} - \|T(F^*\boldsymbol{g}) - T(F^*\boldsymbol{a})\|^2_Y \tag{1.5}$$

and create an iteration process by:

1. Pick $\boldsymbol{g}_0$ and some proper constant $C > 0$

2. Derive a sequence $\{\boldsymbol{g}_k\}_{k=0,1,\ldots}$ by the iteration:

$$\boldsymbol{g}_{k+1} = \arg\min_{\boldsymbol{g}} J^s_\alpha(\boldsymbol{g}, \boldsymbol{g}_k) \qquad k = 0, 1, 2, \ldots$$

As we shall see later on, in order to prove norm convergence of the iterates $\boldsymbol{g}_k$ towards a critical point of $J_\alpha$, we have to restrict ourselves to a class of nonlinear problems for which all of the following three requirements hold true,

$$\boldsymbol{g}_k \xrightarrow{w} \boldsymbol{g} \Longrightarrow T(F^*\boldsymbol{g}_k) \to T(F^*\boldsymbol{g}) \quad , \tag{1.6}$$

$$T'(F^*\boldsymbol{g}_k)^* z \to T'(F^*\boldsymbol{g})^* z \quad , \text{ for all } z \quad , \tag{1.7}$$

$$\|T'(F^*\boldsymbol{g}) - T'(F^*\boldsymbol{g}')\| \leq L B^{1/2} \|\boldsymbol{g} - \boldsymbol{g}'\|_{\ell_2} \quad . \tag{1.8}$$

It may happen that $T$ already meets these conditions as an operator from $X \to Y$. If not, this can be achieved by assuming more regularity of $F^*\boldsymbol{g}$, i.e. changing the domain of $T$ a little. To this end, we assume that there exists a function space $X^s$, and a compact embedding operator $i^s : X^s \to X$. Then we can consider $\tilde{T} = T \circ i^s : X^s \longrightarrow Y$. Lipschitz regularity is preserved. Moreover, if now $F^*\boldsymbol{g}_k \xrightarrow{w} F^*\boldsymbol{g}$ in $X^s$, then $F^*\boldsymbol{g}_k \to F^*\boldsymbol{g}$ in $X$ and, moreover, $\tilde{T}'(F^*\boldsymbol{g}_k) \to \tilde{T}'(F^*\boldsymbol{g})$ in the operator norm. This argument applies to arbitrary nonlinear continuous and Fréchet differentiable operators $T : X \to Y$ with continuous Lipschitz derivative as long as a function space $X^s$ with compact embedding $i^s$ into $X$ is available.

The remaining paper is organized as follows: In Section 2, we explain how the replacement functionals are constructed and discuss the well–posedness of the resulting problem. In Section 3, we derive conditions on the minimizing elements. The main results of the paper are presented in Sections 4 and 5: strong convergence of the iterates towards a critical point and a regularization result in case of an $\ell_1-$ penalty term. We end this paper with Section 6 in which we demonstrate the capabilities of the proposed scheme by solving the nonlinear SPECT problem with respect two classical quadratic and sparsity constraints.

# 2 On the proper definition of the replacement functional

By the definition of $J_\alpha^s$ in (1.5) it is not clear whether the functional is positive definite or even bounded from below. This will be clarified in this section, i.e. we will show that this is the case provided the constant $C$ is chosen properly.

For given $\alpha > 0$ and $\boldsymbol{g}_0$ we define a ball $K_r := \{\boldsymbol{g} \in \ell_2 : \Psi(\boldsymbol{L}\boldsymbol{g}) \leq r\}$, where the radius $r$ is given by

$$r := \frac{\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 + 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_0)}{2\alpha}. \tag{2.1}$$

This obviously ensures $\boldsymbol{g}_0 \in K_r$. Furthermore, we define the constant $C$ by

$$C := 2B \max\left\{ \left( \sup_{\boldsymbol{g} \in K_r} \|T'(F^*\boldsymbol{g})\| \right)^2 , L\sqrt{\|y^\delta - T(F^*\boldsymbol{g}_0)\|^2 + 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_0)} \right\} , \tag{2.2}$$

where $L$ is the Lipschitz constant of the Fréchet derivative of $T$ and $B$ the upper frame bound. We assume that $\boldsymbol{g}_0$ was chosen such that $r < \infty$ and $C < \infty$.

**Lemma 1** *Let $r$ and $C$ be chosen by (2.1), (2.2). Then, for all $\boldsymbol{g} \in K_r$,*

$$C\|\boldsymbol{g} - \boldsymbol{g}_0\|_{\ell_2}^2 - \|T(F^*\boldsymbol{g}) - T(F^*\boldsymbol{g}_0)\|_Y^2 \geq 0 \tag{2.3}$$

*and thus, $J_\alpha(\boldsymbol{g}) \leq J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_0)$.*

*Proof.* By Taylors expansion we have

$$T(F^*\boldsymbol{g} + F^*\boldsymbol{h}) = T(F^*\boldsymbol{g}) + \int_0^1 T'(F^*\boldsymbol{g} + \tau F^*\boldsymbol{h})F^*\boldsymbol{h}\, d\tau$$

and thus we get with $\boldsymbol{h} = \boldsymbol{g}_0 - \boldsymbol{g}$

$$\begin{aligned}
\|T(F^*\boldsymbol{g}) - T(F^*\boldsymbol{g}_0)\|_Y &\leq \int_0^1 \|T'(F^*\boldsymbol{g} + \tau F^*(\boldsymbol{g}_0 - \boldsymbol{g}))\|\|F^*(\boldsymbol{g}_0 - \boldsymbol{g})\|_X d\tau \\
&\leq \sup_{\boldsymbol{g} \in K_r} \|T'(F^*\boldsymbol{g})\|\|F^*(\boldsymbol{g}_0 - \boldsymbol{g})\|_X \\
&\leq \sup_{\boldsymbol{g} \in K_r} \|T'(F^*\boldsymbol{g})\|B^{1/2}\|\boldsymbol{g}_0 - \boldsymbol{g}\|_{\ell_2}
\end{aligned}$$

Consequently, we get for all $\boldsymbol{g} \in K_r$

$$\begin{aligned}
C\|\boldsymbol{g} - \boldsymbol{g}_0\|_{\ell_2}^2 - \|T(F^*\boldsymbol{g}) - T(F^*\boldsymbol{g}_0)\|_Y^2 &\geq C\|\boldsymbol{g} - \boldsymbol{g}_0\|_{\ell_2}^2 - B\left(\sup_{\boldsymbol{g} \in K_r} \|T'(F^*\boldsymbol{g})\|\|\boldsymbol{g} - \boldsymbol{g}_0\|_{\ell_2}\right)^2 \\
&\geq \frac{C}{2}\|\boldsymbol{g} - \boldsymbol{g}_0\|_{\ell_2}^2 \geq 0,
\end{aligned}$$

and the functional $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_0)$ is non–negative for all $\boldsymbol{g} \in K_r$. □

Next, we show that this carries over to all of the iterates:

**Proposition 2** *Let $\boldsymbol{g}_0, \alpha$ be given and $r, C$ be defined by (2.1), (2.2). Then the functionals $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)$ are bounded from below for all $\boldsymbol{g} \in \ell_2$ and all $k \in \mathbb{N}$ and have thus minimizers. For the minimizer $\boldsymbol{g}_{k+1}$ of $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)$ holds $\boldsymbol{g}_{k+1} \in K_r$.*

*Proof.* The proof will be done by induction. For $k = 1$, we show in a first step that $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_0)$ is bounded from below. We have

$$\|y^\delta - T(F^*\boldsymbol{g})\|_Y^2 = \|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 + \|T(F^*\boldsymbol{g}_0) - T(F^*\boldsymbol{g})\|_Y^2 + 2\langle y^\delta - T(F^*\boldsymbol{g}_0), T(F^*\boldsymbol{g}_0) - T(F^*\boldsymbol{g})\rangle_Y. \tag{2.4}$$

Thus,

$$\begin{aligned}
J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_0) - 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}) &= \|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 + 2\langle y^\delta - T(F^*\boldsymbol{g}_0), T(F^*\boldsymbol{g}_0) - T(F^*\boldsymbol{g})\rangle_Y \\
&\qquad + C\|\boldsymbol{g} - \boldsymbol{g}_0\|_{\ell_2}^2 \qquad (2.5) \\
&\geq \|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 - 2\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y\|T(F^*\boldsymbol{g}_0) - T(F^*\boldsymbol{g})\|_Y \\
&\qquad + C\|\boldsymbol{g} - \boldsymbol{g}_0\|_{\ell_2}^2. \qquad (2.6)
\end{aligned}$$

Again by Taylor expansion,

$$\|T(F^*\boldsymbol{g}_0) - T(F^*\boldsymbol{g})\|_Y \le B^{1/2}\|T'(F^*\boldsymbol{g}_0)\|\|\boldsymbol{g}_0 - \boldsymbol{g}\|_{\ell_2} + \frac{BL}{2}\|\boldsymbol{g}_0 - \boldsymbol{g}\|_{\ell_2}^2. \tag{2.7}$$

Now let us assume that $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_0)$ is not bounded from below, e.g. there exists a sequence $\boldsymbol{g}_l$ such that $J_\alpha^s(\boldsymbol{g}_l, \boldsymbol{g}_0) \to -\infty$. This can only hold if $\|T(F^*\boldsymbol{g}_0) - T(F^*\boldsymbol{g}_l)\|_Y \to \infty$, and because of (2.7) follows $\|\boldsymbol{g}_l\|_{\ell_2} \to \infty$ as well. In particular, for $l$ large enough, we derive from (2.7)

$$\|T(F^*\boldsymbol{g}_0) - T(F^*\boldsymbol{g}_l)\|_Y \le BL\|\boldsymbol{g}_0 - \boldsymbol{g}_l\|_{\ell_2}^2,$$

and combining this estimate with (2.6) yields

$$J_\alpha^s(\boldsymbol{g}_l, \boldsymbol{g}_0) - 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_l) \ge \|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 - 2BL\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y\|\boldsymbol{g}_l - \boldsymbol{g}_0\|_{\ell_2}^2 + C\|\boldsymbol{g}_l - \boldsymbol{g}_0\|_{\ell_2}^2.$$

¿From the definition of $C$ in (2.2) follows $2BL\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y \le C$ and thus

$$J_\alpha^s(\boldsymbol{g}_l, \boldsymbol{g}_0) - 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_l) \ge \|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 \ge 0,$$

in contradiction to our assumption $J_\alpha^s(\boldsymbol{g}_l, \boldsymbol{g}_0) \to -\infty$, and thus $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_0)$ is bounded from below. By the same argument, we find $J_\alpha^s(\boldsymbol{g}_l, \boldsymbol{g}_0) \ge 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_l)$ for any sequence $\boldsymbol{g}_l$ with $\|\boldsymbol{g}_l\|_{\ell_2} \to \infty$, and by (1.4) we conclude $J_\alpha^s(\boldsymbol{g}_l, \boldsymbol{g}_0) \to \infty$, i.e. the functional is coercive and has a minimizer $\boldsymbol{g}_1$.

As in (2.6), we get by using (2.7),

$$\begin{aligned}
J_\alpha^s(\boldsymbol{g}_1, \boldsymbol{g}_0) - 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_1) &\ge \|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 + 2\langle y^\delta - T(F^*\boldsymbol{g}_0), T(F^*\boldsymbol{g}_0) - T(F^*\boldsymbol{g}_1)\rangle_Y \\
&\qquad + C\|\boldsymbol{g}_1 - \boldsymbol{g}_0\|_{\ell_2}^2 \\
&\ge \|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 - 2B^{1/2}\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y\|T'(F^*\boldsymbol{g}_0)\|\|\boldsymbol{g}_1 - \boldsymbol{g}_0\|_{\ell_2} \\
&\qquad - BL\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y\|\boldsymbol{g}_1 - \boldsymbol{g}_0\|_{\ell_2}^2 + C\|\boldsymbol{g}_1 - \boldsymbol{g}_0\|_{\ell_2}^2 .
\end{aligned}$$

By (2.2), we have $C/2 \ge BL\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y$, and thus

$$\begin{aligned}
J_\alpha^s(\boldsymbol{g}_1, \boldsymbol{g}_0) - 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_1) &\ge \|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 - 2B^{1/2}\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y\|T'(F^*\boldsymbol{g}_0)\|\|\boldsymbol{g}_1 - \boldsymbol{g}_0\|_{\ell_2} \\
&\qquad + \frac{C}{2}\|\boldsymbol{g}_1 - \boldsymbol{g}_0\|_{\ell_2}^2.
\end{aligned}$$

As $\boldsymbol{g}_0 \in K_r$, it follows from (2.2) that $B^{1/2}\|T'(F^*\boldsymbol{g}_0)\| \le \sqrt{C/2}$ holds, and consequently,

$$\begin{aligned}
J_\alpha^s(\boldsymbol{g}_1, \boldsymbol{g}_0) - 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_1) &\ge \|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 - 2\frac{\sqrt{C}}{\sqrt{2}}\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y\|\boldsymbol{g}_1 - \boldsymbol{g}_0\|_{\ell_2} \\
&\qquad + \frac{C}{2}\|\boldsymbol{g}_1 - \boldsymbol{g}_0\|_{\ell_2}^2 \\
&= \left(\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y - \frac{\sqrt{C}}{\sqrt{2}}\|\boldsymbol{g}_1 - \boldsymbol{g}_0\|_{\ell_2}\right)^2 \ge 0.
\end{aligned}$$

In particular,

$$\begin{aligned}
2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_1) &\le J_\alpha^s(\boldsymbol{g}_1, \boldsymbol{g}_0) = \min_{\boldsymbol{g}} J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_0) \le J_\alpha^s(\boldsymbol{g}_0, \boldsymbol{g}_0) \\
&= \|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 + 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_0) ,
\end{aligned}$$

i.e.

$$\Psi(\boldsymbol{L}\boldsymbol{g}_1) \le \frac{\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 + 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_0)}{2\alpha} = r,$$

and thus $\boldsymbol{g}_1 \in K_r$.

Next, thanks to Lemma 1,

$$C\|\boldsymbol{g}_1 - \boldsymbol{g}_0\|_{\ell_2}^2 - \|T(F^*\boldsymbol{g}_1) - T(F^*\boldsymbol{g}_0)\|_Y^2 \ge 0 \quad \text{and } J_\alpha(\boldsymbol{g}_1) \le J_\alpha^s(\boldsymbol{g}_1, \boldsymbol{g}_0)\ ,$$

and we thus have

$$\|y^\delta - T(F^*\boldsymbol{g}_1)\|_Y^2 \le J_\alpha(\boldsymbol{g}_1) \le J_\alpha^s(\boldsymbol{g}_1, \boldsymbol{g}_0) \le J_\alpha^s(\boldsymbol{g}_0, \boldsymbol{g}_0) \le \|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 + 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_0),$$

and combining this estimate with the definition of $C$ in (2.2) yields

$$2BL\|y^\delta - T(F^*\boldsymbol{g}_1)\|_Y \le 2BL\sqrt{\|y^\delta - T(F^*\boldsymbol{g}_0)\|_Y^2 + 2\alpha\Psi(\boldsymbol{L}\boldsymbol{g}_0)} \le C. \tag{2.8}$$

Assuming now that the following properties hold for all $i = 1, \cdots k-1$:

$$\boldsymbol{g}_i \in K_r \tag{2.9}$$

$$C\|\boldsymbol{g}_i - \boldsymbol{g}_{i-1}\|_{\ell_2}^2 - \|T(F^*\boldsymbol{g}_i) - T(F^*\boldsymbol{g}_{i-1})\|_Y^2 \ge 0 \tag{2.10}$$

$$2BL\|y^\delta - T(F^*\boldsymbol{g}_i)\| \le C, \tag{2.11}$$

where $\boldsymbol{g}_i$ denotes a minimizer of the functional $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_{i-1})$, we may deduce by the same arguments as for $i = 1$ that the functional $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_{k-1})$ has a minimizer and that $\boldsymbol{g}_k \in K_r$. □

As an immediate consequence out of the latter proof we have

**Corollary 3** *The sequences* $\{J_\alpha(\boldsymbol{g}_k)\}_{k=0,1,2,\ldots}$ *and* $\{J_\alpha^s(\boldsymbol{g}_{k+1}, \boldsymbol{g}_k)\}_{k=0,1,2,\ldots}$ *are non-increasing.*

# 3 On the minimization of the replacement functional

In this section, we elaborate necessary conditions for a minimizer of the functional $J_\alpha^s(\boldsymbol{g}, \boldsymbol{a})$.

**Lemma 4** *The necessary condition for a minimum of* $J_\alpha^s(\boldsymbol{g}, \boldsymbol{a})$ *is given by*

$$0 \in -FT'(F^*\boldsymbol{g})^*(y^\delta - T(F^*\boldsymbol{a})) + C\boldsymbol{g} - C\boldsymbol{a} + \alpha\boldsymbol{L}^*\partial\Psi(\boldsymbol{L}\boldsymbol{g})\ \ . \tag{3.1}$$

*Proof.* Introducing the functional $\Theta$ via the relation $\boldsymbol{v} \in \partial\Theta(\boldsymbol{g}) \Leftrightarrow \boldsymbol{L}\boldsymbol{v} \in \partial\Psi(\boldsymbol{L}\boldsymbol{g})$, we obtain in the notion of subgradients ,

$$\partial J_\alpha^s(\boldsymbol{g}, \boldsymbol{a}) = -2FT'(F^*\boldsymbol{g})^*(y^\delta - T(F^*\boldsymbol{a})) + 2C\boldsymbol{g} - 2C\boldsymbol{a} + 2\alpha\partial\Theta(\boldsymbol{g})\ .$$

Consequently, the necessary condition (3.1) follows immediately. □

**Lemma 5** *Let* $M(\boldsymbol{g}, \boldsymbol{a}) := FT'(F^*\boldsymbol{g})^*(y^\delta - T(F^*\boldsymbol{a}))/C + \boldsymbol{a}$. *The necessary condition* (3.1) *can be casted as*

$$\boldsymbol{g} = \frac{\alpha}{C}\boldsymbol{L}^*(I - P_{\mathcal{C}})\left(\frac{C}{\alpha}\boldsymbol{L}M(\boldsymbol{g}, \boldsymbol{a})\right)\ , \tag{3.2}$$

*where* $P_{\mathcal{C}}$ *is an orthogonal projection onto a convex set* $\mathcal{C}$.

Before proving Lemma 5, we will have a closer look to the relation between $\Psi$ and $\mathcal{C}$. We may consider the Fenchel or so–called dual functional of $\Psi$, which we will denote by $\Psi^*$. Since we have assumed $\Psi$ to be a positive and one homogeneous functional, there exists a convex set $\mathcal{C}$ such that $\Psi^*$ is equal to the indicator function $\chi_{\mathcal{C}}$ over $\mathcal{C}$. Moreover, in Hilbert space lore, we have total duality between convex sets and positive and one homogeneous functionals, i.e. $\Psi = (\chi_{\mathcal{C}})^*$.

Let us now prove Lemma 5:

*Proof.* With the shorthand $M(\boldsymbol{g}, \boldsymbol{a})$ for $FT'(F^*\boldsymbol{g})^*(y^\delta - T(F^*\boldsymbol{a}))/C + \boldsymbol{a}$ we may rewrite (3.1),

$$\boldsymbol{L}\frac{M(\boldsymbol{g},\boldsymbol{a}) - \boldsymbol{g}}{\frac{\alpha}{C}} \in \partial\Psi(\boldsymbol{L}\boldsymbol{g}) \ ,$$

and thus, by standard arguments in convex analysis,

$$\frac{C}{\alpha}\boldsymbol{L}\boldsymbol{g} \in \frac{C}{\alpha}\partial\Psi^*\left(\boldsymbol{L}\frac{M(\boldsymbol{g},\boldsymbol{a}) - \boldsymbol{g}}{\frac{\alpha}{C}}\right) \ .$$

In order to have an expression by means of projections, we expand the latter formula as follows

$$\begin{aligned}
\boldsymbol{L}\frac{M(\boldsymbol{g},\boldsymbol{a})}{\frac{\alpha}{C}} &\in \boldsymbol{L}\frac{M(\boldsymbol{g},\boldsymbol{a}) - \boldsymbol{g}}{\frac{\alpha}{C}} + \frac{C}{\alpha}\partial\Psi^*\left(\boldsymbol{L}\frac{M(\boldsymbol{g},\boldsymbol{a}) - \boldsymbol{g}}{\frac{\alpha}{C}}\right) \\
&= \left(I + \frac{C}{\alpha}\partial\Psi^*\right)\left(\boldsymbol{L}\frac{M(\boldsymbol{g},\boldsymbol{a}) - \boldsymbol{g}}{\frac{\alpha}{C}}\right) \ ,
\end{aligned}$$

which is equivalent to

$$\left(I + \frac{C}{\alpha}\partial\Psi^*\right)^{-1}\left(\boldsymbol{L}\frac{M(\boldsymbol{g},\boldsymbol{a})}{\frac{\alpha}{C}}\right) = \boldsymbol{L}\frac{M(\boldsymbol{g},\boldsymbol{a}) - \boldsymbol{g}}{\frac{\alpha}{C}} \ .$$

Again, by standard results in convex analysis, it is known that $\left(I + \frac{C}{\alpha}\partial\Psi^*\right)^{-1}$ is nothing than the orthogonal projection onto a convex set $\mathcal{C}$, and hence the assertion follows,

$$\boldsymbol{g} = \frac{\alpha}{C}\boldsymbol{L}^*(I - P_{\mathcal{C}})\left(\boldsymbol{L}\frac{M(\boldsymbol{g},\boldsymbol{a})}{\frac{\alpha}{C}}\right) \ .$$

□

The latter lemma states that for minimizing (1.5) we need to solve the fixed point equation (3.2). To this end, we introduce the associated fixed point map $\Phi_{\alpha,\mathcal{C}}$ with respect to some $\alpha$ and $\mathcal{C}$, i.e.

$$\Phi_{\alpha,\mathcal{C}}(\boldsymbol{g},\boldsymbol{a}) := \frac{\alpha}{C}\boldsymbol{L}^*(I - P_{\mathcal{C}})\left(\boldsymbol{L}\frac{M(\boldsymbol{g},\boldsymbol{a})}{\frac{\alpha}{C}}\right) .$$

In order to ensure contractivity of $\Phi_{\alpha,\mathcal{C}}$, for some generic $\boldsymbol{a}$, we need to analyze $I - P_{\mathcal{C}}$ beforehand.

**Lemma 6** *The mapping $I - P_{\mathcal{C}}$ is non–expansive.*

To prove this Lemma we need the following two standard properties of convex sets, see [3],

**Lemma 7** *Let $K$ be a closed and convex set in some Hilbert space $H$, then for all $u \in H$ and all $k \in K$ the inequality $\langle u - P_K u, k - P_K u\rangle \leq 0$ holds true.*

**Lemma 8** *Let $K$ be a closed and convex set, then for all $u, v \in H$ the inequality*

$$\|u - v - (P_K u - P_K v)\| \leq \|u - v\|$$

*holds true.*

Thanks to Lemma 8 we still have assured Lemma 6, and with Lemma 6 at hand we are able to clarify whether $\Phi_{\alpha,\mathcal{C}}(\cdot, \boldsymbol{a})$ is a contraction operator.

**Lemma 9** *The operator $\Phi_{\alpha,\mathcal{C}}(\cdot, \boldsymbol{a})$ is a contraction, i.e.*

$$\|\Phi_{\alpha,\mathcal{C}}(\boldsymbol{g}, \boldsymbol{a}) - \Phi_{\alpha,\mathcal{C}}(\tilde{\boldsymbol{g}}, \boldsymbol{a})\|_{\ell_2} \leq q\|\boldsymbol{g} - \tilde{\boldsymbol{g}}\|_{\ell_2} \quad \text{if} \quad q := \frac{BL}{C}\sqrt{J_\alpha(\boldsymbol{a})} < 1 \ .$$

*Proof.* We have by Lemma 6 and the Lipschitz–continuity of $T'$

$$\begin{aligned}
\|\Phi_{\alpha,\mathcal{C}}(\boldsymbol{g}, \boldsymbol{a}) - \Phi_{\alpha,\mathcal{C}}(\tilde{\boldsymbol{g}}, \boldsymbol{a})\|_{\ell_2} &= \frac{\alpha}{C}\left\|(I - P_{\mathcal{C}})\left(\boldsymbol{L}\frac{M(\boldsymbol{g}, \boldsymbol{a})}{\frac{\alpha}{C}}\right) - (I - P_{\mathcal{C}})\left(\boldsymbol{L}\frac{M(\tilde{\boldsymbol{g}}, \boldsymbol{a})}{\frac{\alpha}{C}}\right)\right\|_{\ell_2} \\
&\leq \|M(\boldsymbol{g}, \boldsymbol{a}) - M(\tilde{\boldsymbol{g}}, \boldsymbol{a})\|_{\ell_2} \\
&\leq \frac{\sqrt{B}}{C}\|T'(F^*\boldsymbol{g}) - T'(F^*\tilde{\boldsymbol{g}})\|\|y^\delta - T(F^*\boldsymbol{a})\|_Y \\
&\leq \frac{BL}{C}\sqrt{J_\alpha(\boldsymbol{a})}\|\boldsymbol{g} - \tilde{\boldsymbol{g}}\|_{\ell_2}
\end{aligned}$$

and the assertion follows. □

**Proposition 10** *The fixed point map $\Phi_{\alpha,\mathcal{C}}(\boldsymbol{g}, \boldsymbol{g}_k)$ to solve the fixed point equation (3.2) is for all $k = 0, 1, 2, \dots$ and all $\alpha \geq 0$ and $\mathcal{C}$ a contraction.*

*Proof.* By the definition of $C$ in (2.2) and Lemma 9 (setting $\boldsymbol{a} = \boldsymbol{g}_0$), we deduce that $\Phi_{\alpha,\mathcal{C}}(\boldsymbol{g}, \boldsymbol{g}_0)$ is a contraction with

$$q = \frac{BL}{C}\sqrt{J_\alpha(\boldsymbol{g}_0)} \leq \frac{1}{2} < 1.$$

With the help of Corollary 3, we complete the proof

$$\begin{aligned}
\|\Phi_{\alpha,\mathcal{C}}(\boldsymbol{g}, \boldsymbol{g}_k) - \Phi_{\alpha,\mathcal{C}}(\tilde{\boldsymbol{g}}, \boldsymbol{g}_k)\|_{\ell_2} &\leq \frac{BL}{C}\sqrt{J_\alpha(\boldsymbol{g}_k)}\|\boldsymbol{g} - \tilde{\boldsymbol{g}}\|_{\ell_2} \\
&\leq \dots \leq \frac{BL}{C}\sqrt{J_\alpha(\boldsymbol{g}_0)}\|\boldsymbol{g} - \tilde{\boldsymbol{g}}\|_{\ell_2} \leq \frac{1}{2}\|\boldsymbol{g} - \tilde{\boldsymbol{g}}\|_{\ell_2}.
\end{aligned}$$

□

Up to here, we do know that our fixed point iteration converges towards a critical point of $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)$.

**Proposition 11** *The necessary equation* (3.2) *for a minimum of the functional* $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)$ *has a unique fixed point, and the fixed point iteration converges towards the minimizer.*

*Proof.* To verify this assertion, we have to investigate the Taylor expansion of $J_\alpha^s$ more closely. By Taylor's expansion for $T$ and the Lipschitz–continuity of $T'$ we get

$$T(F^*\boldsymbol{g} + F^*\boldsymbol{h}) = T(F^*\boldsymbol{g}) + T'(F^*\boldsymbol{g})F^*\boldsymbol{h} + R(F^*\boldsymbol{g}, F^*\boldsymbol{h}) \tag{3.3}$$

with

$$\|R(F^*\boldsymbol{g}, F^*\boldsymbol{h})\|_Y \leq \frac{BL}{2}\|\boldsymbol{h}\|_{\ell_2}^2 \ . \tag{3.4}$$

Next, we observe,

$$\begin{aligned} J_\alpha^s(\boldsymbol{g}+\boldsymbol{h}, \boldsymbol{g}_k) - J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k) \ &= \ \partial J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)\boldsymbol{h} + C\|\boldsymbol{h}\|_{\ell_2}^2 + 2\alpha\{\Theta(\boldsymbol{g}+\boldsymbol{h}) - \Theta(\boldsymbol{g}) - \partial\Theta(\boldsymbol{g})\boldsymbol{h}\} \\ &\qquad -2\langle y^\delta - T(F^*\boldsymbol{g}_k), R(F^*\boldsymbol{g}, F^*\boldsymbol{h})\rangle_Y \\ &\geq \ \partial J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)\boldsymbol{h} + C\|\boldsymbol{h}\|_{\ell_2}^2 + 2\alpha\{\Theta(\boldsymbol{g}+\boldsymbol{h}) - \Theta(\boldsymbol{g}) - \partial\Theta(\boldsymbol{g})\boldsymbol{h}\} \\ &\qquad -2\|y^\delta - T(F^*\boldsymbol{g}_k)\|_{\ell_2}\frac{BL}{2}\|\boldsymbol{h}\|_{\ell_2}^2 \\ &\geq \ \partial J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)\boldsymbol{h} + \frac{C}{2}\|\boldsymbol{h}\|_{\ell_2}^2 + 2\alpha\{\Theta(\boldsymbol{g}+\boldsymbol{h}) - \Theta(\boldsymbol{g}) - \partial\Theta(\boldsymbol{g})\boldsymbol{h}\}. \end{aligned}$$

Assuming $\boldsymbol{g}$ is a critical point, i.e. for all $\boldsymbol{v} \in \partial J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)$ and all $\boldsymbol{h} \in \ell_2$ one has $\langle \boldsymbol{v}, \boldsymbol{h}\rangle_{\ell_2} = 0$ or equivalently written $\partial J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)\boldsymbol{h} = 0$, we have

$$J_\alpha^s(\boldsymbol{g}+\boldsymbol{h}, \boldsymbol{g}_k) - J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k) \geq \frac{C}{2}\|\boldsymbol{h}\|_{\ell_2}^2 + 2\alpha\{\Theta(\boldsymbol{g}+\boldsymbol{h}) - \Theta(\boldsymbol{g}) - \partial\Theta(\boldsymbol{g})\boldsymbol{h}\} \ .$$

Now via the definition of subgradients: an element $\boldsymbol{v} \in \ell_2$ belongs to $\partial\Theta(\boldsymbol{g})$ if and only if for all $\boldsymbol{x} \in \ell_2$,

$$\Theta(\boldsymbol{g}) + \langle \boldsymbol{v}, \boldsymbol{x} - \boldsymbol{g}\rangle_{\ell_2} \leq \Theta(\boldsymbol{x}) \ ,$$

and, in particular for $\boldsymbol{x} = \boldsymbol{g} + \boldsymbol{h}$, this yields for all $\boldsymbol{v} \in \partial\Theta(\boldsymbol{g})$ and all $\boldsymbol{h} \in \ell_2$,

$$\Theta(\boldsymbol{g}) + \langle \boldsymbol{v}, \boldsymbol{h}\rangle_{\ell_2} \leq \Theta(\boldsymbol{g}+\boldsymbol{h}) \ \text{ or, equivalently, } \ 0 \leq \Theta(\boldsymbol{g}+\boldsymbol{h}) - \Theta(\boldsymbol{g}) - \partial\Theta(\boldsymbol{g})\boldsymbol{h} \ .$$

Consequently,

$$J_\alpha^s(\boldsymbol{g}+\boldsymbol{h}, \boldsymbol{g}_k) - J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k) \geq \frac{C}{2}\|\boldsymbol{h}\|_{\ell_2}^2 \ ,$$

and thus every critical point is a global minimizer of $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)$, and, again by the latter inequality, there exists only one global minimizer. □

By assuming more regularity on $T$, the latter statement can be improved a little:

**Proposition 12** *Let $T$ be a twice continuously differentiable operator. Then the functional* $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)$ *is strictly convex.*

*Proof.* Since the non–convex part of $J_\alpha^s$ is the discrepancy $\|y^\delta - T(F^*\boldsymbol{g})\|_Y^2$, it remains to show that

$$J^d(\boldsymbol{g}) := \|y^\delta - T(F^*\boldsymbol{g})\|_Y^2 + C\|\boldsymbol{g} - \boldsymbol{g}_k\|_{\ell_2}^2 - \|T(F^*\boldsymbol{g}) - T(F^*\boldsymbol{g}_k)\|_Y^2 \tag{3.5}$$

is strictly convex in $\boldsymbol{g}$, i.e. we have to show that

$$J^d((1-\lambda)\boldsymbol{g}_1 + \lambda\boldsymbol{g}_2) < (1-\lambda)J^d(\boldsymbol{g}_1) + \lambda J^d(\boldsymbol{g}_2)$$

holds for $\lambda \in (0,1)$ and arbitrary $\boldsymbol{g}_1, \boldsymbol{g}_2 \in \ell_2$. At first, we express $J^d$ by its Taylor expansion,

$$J^d(\boldsymbol{g} + \boldsymbol{h}) = J^d(\boldsymbol{g}) + DJ^d(\boldsymbol{g})\boldsymbol{h} + r(\boldsymbol{g}, \boldsymbol{h}) \ , \tag{3.6}$$

where

$$r(\boldsymbol{g}, \boldsymbol{h}) := -2\langle y^\delta - T(F^*\boldsymbol{g}_k), R(F^*\boldsymbol{g}, F^*\boldsymbol{h})\rangle_Y + C\|\boldsymbol{h}\|_{\ell_2}^2 \ . \tag{3.7}$$

We have

$$\begin{aligned} J^d((1-\lambda)\boldsymbol{g}_1 + \lambda\boldsymbol{g}_2)) &= J^d(\boldsymbol{g}_1 + \lambda(\boldsymbol{g}_2 - \boldsymbol{g}_1)) = J^d(\boldsymbol{g}_2 + (1-\lambda)(\boldsymbol{g}_1 - \boldsymbol{g}_2)) \\ &= (1-\lambda)J^d(\boldsymbol{g}_1 + \lambda(\boldsymbol{g}_2 - \boldsymbol{g}_1)) + \lambda J^d(\boldsymbol{g}_2 + (1-\lambda)(\boldsymbol{g}_1 - \boldsymbol{g}_2)) \end{aligned} \tag{3.8}$$

and with

$$\begin{aligned} J^d(\boldsymbol{g}_1 + \lambda(\boldsymbol{g}_2 - \boldsymbol{g}_1)) &= J^d(\boldsymbol{g}_1) + \lambda DJ^d(\boldsymbol{g}_1)(\boldsymbol{g}_2 - \boldsymbol{g}_1) + r(\boldsymbol{g}_1, \lambda(\boldsymbol{g}_2 - \boldsymbol{g}_1)) \\ J^d(\boldsymbol{g}_2 + (1-\lambda)(\boldsymbol{g}_1 - \boldsymbol{g}_2)) &= J^d(\boldsymbol{g}_2) + (1-\lambda)DJ^d(\boldsymbol{g}_2)(\boldsymbol{g}_1 - \boldsymbol{g}_2) + r(\boldsymbol{g}_2, (1-\lambda)(\boldsymbol{g}_1 - \boldsymbol{g}_2)) \end{aligned}$$

we obtain

$$\begin{aligned} J^d((1-\lambda)\boldsymbol{g}_1 + \lambda\boldsymbol{g}_2)) &= (1-\lambda)J^d(\boldsymbol{g}_1) + \lambda J^d(\boldsymbol{g}_2) + \lambda(1-\lambda)\left[DJ^d(\boldsymbol{g}_1) - DJ^d(\boldsymbol{g}_2)\right](\boldsymbol{g}_2 - \boldsymbol{g}_1) \\ &\quad + (1-\lambda)r(\boldsymbol{g}_1, \lambda(\boldsymbol{g}_2 - \boldsymbol{g}_1)) + \lambda r(\boldsymbol{g}_2, (1-\lambda)(\boldsymbol{g}_1 - \boldsymbol{g}_2)) \ . \end{aligned}$$

Thus, $J_\alpha^s$ is strictly convex if for all $\lambda \in (0,1)$,

$$\begin{aligned} D(\boldsymbol{g}_1, \boldsymbol{g}_2, \lambda) &:= \lambda(1-\lambda)\left[DJ^d(\boldsymbol{g}_1) - DJ^d(\boldsymbol{g}_2)\right](\boldsymbol{g}_2 - \boldsymbol{g}_1) \\ &\quad + (1-\lambda)r(\boldsymbol{g}_1, \lambda(\boldsymbol{g}_2 - \boldsymbol{g}_1)) + \lambda r(\boldsymbol{g}_2, (1-\lambda)(\boldsymbol{g}_1 - \boldsymbol{g}_2)) < 0 \ . \end{aligned}$$

We have

$$\begin{aligned} \left[DJ^d(\boldsymbol{g}_1) - DJ^d(\boldsymbol{g}_2)\right](\boldsymbol{g}_2 - \boldsymbol{g}_1) &= -2C\|\boldsymbol{g}_2 - \boldsymbol{g}_1\|_{\ell_2}^2 \\ &\quad -2\langle y^\delta - T(F^*\boldsymbol{g}_k), (T'(F^*\boldsymbol{g}_1) - T'(F^*\boldsymbol{g}_2))F^*(\boldsymbol{g}_2 - \boldsymbol{g}_1)\rangle_Y. \end{aligned}$$

As $T$ is twice continuously Fréchet differentiable, it is

$$T'(F^*\boldsymbol{g}_1) = T'(F^*\boldsymbol{g}_2) + \int_0^1 T''(F^*\boldsymbol{g}_2 + \tau F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))(F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2), \cdot)\, d\tau$$

and thus,

$$\begin{aligned} &\left[DJ^d(\boldsymbol{g}_1) - DJ^d(\boldsymbol{g}_2)\right](\boldsymbol{g}_2 - \boldsymbol{g}_1) = \\ &\qquad -2C\|\boldsymbol{g}_2 - \boldsymbol{g}_1\|_{\ell_2}^2 + 2\langle y^\delta - T(F^*\boldsymbol{g}_k), \int_0^1 T''(F^*\boldsymbol{g}_2 + \tau F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))(F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))^2 d\tau\rangle, \end{aligned} \tag{3.9}$$

where we have used the shorthand $T''(\cdot)(\cdot,\cdot) = T''(\cdot)(\cdot)^2$. Again, as $T$ is twice continuously Fréchet-differentiable, the function $R(F^*\boldsymbol{g}, F^*\boldsymbol{h})$ in (3.7) is given by

$$R(F^*\boldsymbol{g}, F^*\boldsymbol{h}) = \int_0^1 (1-\tau)T''(F^*\boldsymbol{g} + \tau F^*\boldsymbol{h})(F^*\boldsymbol{h})^2\, d\tau \ ,$$

and thus we obtain

$$\begin{aligned} R(F^*\boldsymbol{g}_1, \lambda F^*(\boldsymbol{g}_2 - \boldsymbol{g}_1)) &= \lambda^2 \int_0^1 (1-\tau)T''(F^*\boldsymbol{g}_1 + \tau\lambda F^*(\boldsymbol{g}_2 - \boldsymbol{g}_1))(F^*(\boldsymbol{g}_2 - \boldsymbol{g}_1))^2\, d\tau \\ &= \int_{1-\lambda}^1 (\tau - (1-\lambda))T''(F^*\boldsymbol{g}_2 + \tau F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))(F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))^2\, d\tau \end{aligned} \tag{3.10}$$

and in the same way

$$R(F^*\boldsymbol{g}_2, (1-\lambda)F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2)) = \int_0^{1-\lambda} (1-\lambda-\tau)T''(F^*\boldsymbol{g}_2 + \tau F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))(F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))^2\, d\tau \ . \tag{3.11}$$

Combining definition (3.7) and equations (3.9), (3.10) and (3.11) yields

$$D(\boldsymbol{g}_1, \boldsymbol{g}_2, \lambda) = -\lambda(1-\lambda)C\|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{\ell_2}^2 + 2\langle y^\delta - T(F^*\boldsymbol{g}_k), f(\boldsymbol{g}_1, \boldsymbol{g}_2, \lambda)\rangle_Y \ , \tag{3.12}$$

where

$$\begin{aligned} f(\boldsymbol{g}_1, \boldsymbol{g}_2, \lambda) \ &:= \ \lambda(1-\lambda)\int_0^1 T''(F^*\boldsymbol{g}_2 + \tau F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))(F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))^2\, d\tau \\ &\quad -(1-\lambda)\int_{1-\lambda}^1 (\tau - (1-\lambda))T''(F^*\boldsymbol{g}_2 + \tau F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))(F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))^2\, d\tau \\ &\quad -\lambda \int_0^{1-\lambda} (1-\lambda-\tau)T''(F^*\boldsymbol{g}_2 + \tau F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))(F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2)^2\, d\tau \ . \end{aligned}$$

The functional $f(\boldsymbol{g}_1, \boldsymbol{g}_2, \lambda)$ can now be recasted as follows

$$\begin{aligned} f(x_1, x_2, \lambda) \ &= \ \lambda \int_0^{1-\lambda} \tau T''(F^*\boldsymbol{g}_2 + \tau F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))(F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))^2\, d\tau \\ &\quad +(1-\lambda)\int_{1-\lambda}^1 (1-\tau)T''(F^*\boldsymbol{g}_2 + \tau F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))(F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))^2\, d\tau. \end{aligned}$$

In order to estimate $\|f(\boldsymbol{g}_1, \boldsymbol{g}_2, \lambda)\|_Y$ it is necessary to estimate the integrals separately. Due to the Lipschitz–continuity of the first derivative, the second derivative can be globally estimated by $L$, and it follows,

$$\begin{aligned}
\lambda \left\| \int_0^{1-\lambda} \tau T''(F^*\boldsymbol{g}_2 + \tau F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))(F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))^2 \, d\tau \right\|_Y &\leq \lambda \frac{(1-\lambda)^2}{2} BL \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{\ell_2}^2 \, , \\
(1-\lambda) \left\| \int_{1-\lambda}^{1} (1-\tau) T''(F^*\boldsymbol{g}_2 + \tau F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))(F^*(\boldsymbol{g}_1 - \boldsymbol{g}_2))^2 \, d\tau \right\|_Y &\leq (1-\lambda) \frac{\lambda^2}{2} BL \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{\ell_2}^2
\end{aligned}$$

and thus

$$\|f(\boldsymbol{g}_1, \boldsymbol{g}_2, \lambda)\|_Y \leq \frac{\lambda(1-\lambda)}{2} BL \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{\ell_2}^2 \, . \tag{3.13}$$

Combining (3.12) and (3.13) yields for $\lambda \in (0, 1)$

$$\begin{aligned}
D(\boldsymbol{g}_1, \boldsymbol{g}_2, \lambda) &\leq -\lambda(1-\lambda) C \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{\ell_2}^2 + 2\|y^\delta - T(F^*\boldsymbol{g}_k)\|_Y \|f(\boldsymbol{g}_1, \boldsymbol{g}_2, \lambda)\|_Y \\
&\leq -\lambda(1-\lambda) C \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{\ell_2}^2 + \frac{\lambda(1-\lambda)}{2} 2BL \|y^\delta - T(F^*\boldsymbol{g}_k)\| \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{\ell_2}^2 \\
&\leq -\lambda(1-\lambda) \frac{C}{2} \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{\ell_2}^2 < 0 \, ,
\end{aligned}$$

and thus the functional is strictly convex. □

# 4 Convergence properties of the iteration

Within this section we discuss convergence properties of the proposed scheme, i.e. we aim to show that the sequence of iterates $\{\boldsymbol{g}_k\}$ converges strongly towards a critical point of $J_\alpha$, at least.

**Lemma 13** *The sequence of iterates $\{\boldsymbol{g}_k\}$ has a weakly convergent subsequence.*

*Proof.* This is an immediate consequence of Proposition 2, in which we have shown that for $k = 0, 1, 2, \ldots$ the iterates $\boldsymbol{g}_k$ are contained in $K_r$, i.e. $\|\boldsymbol{g}_k\|_{\ell_2} \leq r$. Since the iterates are uniformly bounded, we deduce that there exists at least one accumulation point $\boldsymbol{g}_\alpha^\star$ with $\boldsymbol{g}_{k_l} \xrightarrow{w} \boldsymbol{g}_\alpha^\star$, where $\boldsymbol{g}_{k_l}$ denotes a subsequence of $\boldsymbol{g}_k$. □

**Lemma 14** *For the iterates $\boldsymbol{g}_k$ holds* $\lim_{k\to\infty} \|\boldsymbol{g}_{k+1} - \boldsymbol{g}_k\|_{\ell_2} = 0$.

*Proof.* With the help of Corollary 3, we observe that

$$\begin{aligned} 0 &\le \sum_{k=0}^{N} \left\{ C\|\boldsymbol{g}_{k+1} - \boldsymbol{g}_k\|_{\ell_2}^2 - \|T(F^*\boldsymbol{g}_{k+1}) - T(F^*\boldsymbol{g}_k)\|_Y^2 \right\} \\ &= \sum_{k=0}^{N} \left\{ J_\alpha^s(\boldsymbol{g}_{k+1}, \boldsymbol{g}_k) - J_\alpha(\boldsymbol{g}_{k+1}) \right\} \le \sum_{k=0}^{N} \left\{ J_\alpha(\boldsymbol{g}_k) - J_\alpha(\boldsymbol{g}_{k+1}) \right\} \\ &= J_\alpha(\boldsymbol{g}_0) - J_\alpha(\boldsymbol{g}_{N+1}) \le J_\alpha(\boldsymbol{g}_0) , \end{aligned}$$

i.e. the finite sums are uniformly bounded (independent on $N$). Now, by the Taylor expansion of $T$, we have

$$\|T(F^*\boldsymbol{g}_{k+1}) - T(F^*\boldsymbol{g}_k)\|_Y^2 \le \frac{C}{2}\|\boldsymbol{g}_{k+1} - \boldsymbol{g}_k\|_{\ell_2}^2 ,$$

and thus

$$0 \le \frac{C}{2}\|\boldsymbol{g}_{k+1} - \boldsymbol{g}_k\|_{\ell_2}^2 \le C\|\boldsymbol{g}_{k+1} - \boldsymbol{g}_k\|_{\ell_2}^2 - \|T(F^*\boldsymbol{g}_{k+1}) - T(F^*\boldsymbol{g}_k)\|_Y^2 \longrightarrow 0$$

as $k \to \infty$ and the assertion follows. $\square$

To obtain a convergence result, we need the following preliminary lemmatas. They state properties involving the general constraint $\Theta$. However, when showing strong convergence we have to restrict ourselves to a the class of constraints of weighted $\ell_p$ norms.

**Lemma 15** *Let $\Theta$ be a convex and weakly lower semi–continuous functional. For sequences $\boldsymbol{v}_k \to \boldsymbol{v}$ and $\boldsymbol{g}_k \overset{w}{\to} \boldsymbol{g}$, assume $\boldsymbol{v}_k \in \partial\Theta(\boldsymbol{g}_k)$ for all $k \in \mathbb{N}$. Then, $\boldsymbol{v} \in \partial\Theta(\boldsymbol{g})$.*

*Proof.* First, we observe for fixed $\boldsymbol{x} \in \ell_2$,

$$\lim_{k\to\infty} \langle \boldsymbol{v}_k, \boldsymbol{x} - \boldsymbol{g}_k \rangle_{\ell_2} = \lim_{k\to\infty} \langle \boldsymbol{v}_k - \boldsymbol{v}, \boldsymbol{x} - \boldsymbol{g}_k \rangle_{\ell_2} + \lim_{k\to\infty} \langle \boldsymbol{v}, \boldsymbol{x} - \boldsymbol{g}_k \rangle_{\ell_2} ,$$

and because of $|\langle \boldsymbol{v}_k - \boldsymbol{v}, \boldsymbol{x} - \boldsymbol{g}_k \rangle_{\ell_2}| \le \mathit{const} \cdot \|\boldsymbol{v}_k - \boldsymbol{v}\| \to 0$, it follows from the weak convergence of $\{\boldsymbol{g}_k\}$ that

$$\lim_{k\to\infty} \langle \boldsymbol{v}_k, \boldsymbol{x} - \boldsymbol{g}_k \rangle_{\ell_2} = \langle \boldsymbol{v}, \boldsymbol{x} - \boldsymbol{g} \rangle_{\ell_2} .$$

By definition we have $\boldsymbol{v} \in \partial\Theta(\boldsymbol{g})$ if and only if the inequality $\Theta(\boldsymbol{x}) \ge \Theta(\boldsymbol{g}) + \langle \boldsymbol{v}, \boldsymbol{x} - \boldsymbol{g} \rangle_{\ell_2}$ holds true for all $\boldsymbol{x} \in \ell_2$. Since $\{\boldsymbol{v}_k\}$ converges strongly and $\{\boldsymbol{g}_k\}$ weakly, and by the lower semi–continuity of $\Theta$, and, moreover, by the assumption $\boldsymbol{v}_k \in \partial\Theta(\boldsymbol{g}_k)$ (i.e. for all $\boldsymbol{x} \in \ell_2$ the inequality $\Theta(\boldsymbol{x}) \ge \Theta(\boldsymbol{g}_k) + \langle \boldsymbol{v}_k, \boldsymbol{x} - \boldsymbol{g}_k \rangle_{\ell_2}$ holds true) we deduce

$$\begin{aligned} \Theta(\boldsymbol{x}) &\ge \liminf_{k\to\infty} \Theta(\boldsymbol{g}_k) + \liminf_{k\to\infty} \langle \boldsymbol{v}_k, \boldsymbol{x} - \boldsymbol{g}_k \rangle_{\ell_2} \\ &\ge \Theta(\boldsymbol{g}) + \langle \boldsymbol{v}, \boldsymbol{x} - \boldsymbol{g} \rangle_{\ell_2} \end{aligned}$$

for all $\boldsymbol{x}$, and thus $\boldsymbol{v} \in \partial\Theta(\boldsymbol{g})$. $\square$

**Lemma 16** *Every subsequence of $\boldsymbol{g}_k$ has a weakly convergent subsequence $\boldsymbol{g}_{k_l}$ with weak limit $\boldsymbol{g}_\alpha^\star$ that satisfies the necessary condition for a minimizer of $J_\alpha$,*

$$FT'(F^*\boldsymbol{g}_\alpha^\star)^*(y^\delta - T(F^*\boldsymbol{g}_\alpha^\star)) \in \alpha\partial\Theta(\boldsymbol{g}_\alpha^\star) \ . \tag{4.1}$$

*Proof.* According to Lemma 4, the minimizer $\boldsymbol{g}_{k+1}$ of $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)$ fulfills

$$0 \in FT'(F^*\boldsymbol{g}_{k+1})^*(y^\delta - T(F^*\boldsymbol{g}_k)) - C\boldsymbol{g}_{k+1} + C\boldsymbol{g}_k - \alpha\partial\Theta(\boldsymbol{g}_{k+1}).$$

Thus, by defining

$$\begin{aligned} \boldsymbol{v}_{k+1} \ := \ & -\frac{1}{\alpha}\big(C\boldsymbol{g}_{k+1} - C\boldsymbol{g}_k - FT'(F^*\boldsymbol{g}_{k+1})^*(y^\delta - T(F^*\boldsymbol{g}_{k+1})) \\ & -FT'(F^*\boldsymbol{g}_{k+1})^*(T(F^*\boldsymbol{g}_{k+1}) - T(F^*\boldsymbol{g}_k))\big) \end{aligned}$$

we observe

$$\boldsymbol{v}_{k+1} \in \partial\Theta(\boldsymbol{g}_{k+1}) \ .$$

In order to derive the limit of $\{\boldsymbol{v}_k\}$, we apply at first Lemma 14, thus we have

$$\|FT'(F^*\boldsymbol{g}_{k+1})^*(T(F^*\boldsymbol{g}_{k+1}) - T(F^*\boldsymbol{g}_k))\|_Y \leq \sqrt{C/2}\|\boldsymbol{g}_{k+1} - \boldsymbol{g}_k\|_{\ell_2} \to 0 \ .$$

To control the remaining term, we take advantage of Lemma 13, i.e. there exists a subsequence $\{\boldsymbol{g}_{k_l}\} \subset \{\boldsymbol{g}_k\}$ that converges weakly towards its weak limit $\boldsymbol{g}_\alpha^\star$. By the following recast

$$\begin{aligned} & FT'(F^*\boldsymbol{g}_{k_l})^*(y^\delta - T(F^*\boldsymbol{g}_{k_l})) = \\ & \quad FT'(F^*\boldsymbol{g}_{k_l})^*(y^\delta - T(F^*\boldsymbol{g}_\alpha^\star)) + FT'(F^*\boldsymbol{g}_{k_l})^*(T(F^*\boldsymbol{g}_\alpha^\star) - T(F^*\boldsymbol{g}_{k_l})) \ , \end{aligned}$$

we find that

$$\|FT'(F^*\boldsymbol{g}_{k_l})^*(T(F^*\boldsymbol{g}_\alpha^\star - T(F^*\boldsymbol{g}_{k_l}))\|_{\ell_2} \leq \sqrt{C/2}\|T(F^*\boldsymbol{g}_\alpha^\star) - T(F^*\boldsymbol{g}_{k_l})\|_{\ell_2} \overset{(1.6)}{\to} 0$$

and, moreover by assumption (1.7),

$$FT'(F^*\boldsymbol{g}_{k_l})^*(y^\delta - T(F^*\boldsymbol{g}_\alpha^\star)) \to FT'(F^*\boldsymbol{g}_\alpha^\star)^*(y^\delta - T(F^*\boldsymbol{g}_\alpha^\star)).$$

Consequently, we obtain

$$\lim_{l\to\infty} FT'(F^*\boldsymbol{g}_{k_l})^*(y^\delta - T(F^*\boldsymbol{g}_{k_l})) = FT'(F^*\boldsymbol{g}_\alpha^\star)^*(y^\delta - T(F^*\boldsymbol{g}_\alpha^\star)) \ , \tag{4.2}$$

and hence

$$\lim_l \boldsymbol{v}_{k_l} = FT'(F^*\boldsymbol{g}_\alpha^\star)^*(y^\delta - T(F^*\boldsymbol{g}_\alpha^\star)) =: \boldsymbol{v} \ . \tag{4.3}$$

As we have additionally $\boldsymbol{g}_{k_l} \overset{w}{\to} \boldsymbol{g}_\alpha^\star$, we conclude from Lemma 15 (applied to $\{\boldsymbol{v}_{k_l}\}$) that $\boldsymbol{v} \in \partial\Theta(\boldsymbol{g}_\alpha^\star)$, which completes the proof. □

**Lemma 17** *Let $\{\boldsymbol{g}_{k_l}\} \subset \{\boldsymbol{g}_k\}$ with $\boldsymbol{g}_{k_l} \overset{w}{\to} \boldsymbol{g}_\alpha^\star$. Then,* $\lim_{l\to\infty} \Theta(\boldsymbol{g}_{k_l}) = \Theta(\boldsymbol{g}_\alpha^\star)$

*Proof.* Since $\Theta$ is weakly semi–continuous, we have

$$\Theta(\boldsymbol{g}_\alpha^\star) \le \liminf_{l\to\infty}\Theta(\boldsymbol{g}_{k_l}). \tag{4.4}$$

On the other hand, with the notation of the previous proof, we have seen that $\boldsymbol{v}_{k_l} \in \partial\Theta(\boldsymbol{g}_{k_l})$, which means that for all $\boldsymbol{x} \in \ell_2$, $\Theta(\boldsymbol{x}) \ge \Theta(\boldsymbol{g}_{k_l}) + \langle \boldsymbol{v}_{k_l}, \boldsymbol{x} - \boldsymbol{g}_{k_l}\rangle$. Selecting $\boldsymbol{x} = \boldsymbol{g}_\alpha^\star$, we have

$$\Theta(\boldsymbol{g}_\alpha^\star) \ge \Theta(\boldsymbol{g}_{k_l}) + \langle \boldsymbol{v}_{k_l}, \boldsymbol{g}_\alpha^\star - \boldsymbol{g}_{k_l}\rangle$$

and as $\boldsymbol{v}_{k_l} \to \boldsymbol{v}$, $\boldsymbol{g}_{k_l} \overset{w}{\to} \boldsymbol{g}_\alpha^\star$ it follows $\langle \boldsymbol{v}_{k_l}, \boldsymbol{g}_\alpha^\star - \boldsymbol{g}_{k_l}\rangle \to 0$ and consequently,

$$\Theta(\boldsymbol{g}_\alpha^\star) \ge \limsup_{l\to\infty}\Theta(\boldsymbol{g}_{k_l}) \ . \tag{4.5}$$

Combining (4.4), (4.5) yields the assertion. □

In next theorem we show that with the help of the previous lemmatas and restricting to weighted $\ell_p$ norms, we can achieve strong convergence of the subsequence $\{\boldsymbol{g}_{k_l}\}$. For simplicity we have chosen $\boldsymbol{L}$ to be the identity. However, the theorem can also be shown for isometrically tranformed $\boldsymbol{g}_{k_l}$'s.

**Theorem 18** *Let $\{\boldsymbol{g}_{k_l}\} \subset \{\boldsymbol{g}_k\}$ with $\boldsymbol{g}_{k_l} \overset{w}{\to} \boldsymbol{g}_\alpha^\star$. Assume, moreover, that*

$$\Theta(\boldsymbol{g}) = \Psi(\boldsymbol{g}) = \left(\sum_j \alpha_j |(\boldsymbol{g})_j|^p\right)^{1/p} \tag{4.6}$$

*with $\alpha_j \ge 1$ and $1 \le p \le 2$. Then the subsequence $\{\boldsymbol{g}_{k_l}\}$ converges also in norm.*

*Proof.* Let us first assume for all $l$ that $(\boldsymbol{g}_{k_l})_j \le 1$. Setting $D = \left|\sum_j^\infty |(\boldsymbol{g}_{k_l})_j|^2 - \sum_j^\infty |(\boldsymbol{g}_\alpha^\star)_j|^2\right|$, we have

$$D \le \left|\sum_j^N |(\boldsymbol{g}_{k_l})_j|^2 - |(\boldsymbol{g}_\alpha^\star)_j|^2\right| + \sum_{N+1}^\infty |(\boldsymbol{g}_{k_l})_j|^2 + \sum_{N+1}^\infty |(\boldsymbol{g}_\alpha^\star)_j|^2 \tag{4.7}$$

For fixed $0 < \varepsilon$, we choose $N$ such that

$$\sum_{N+1}^\infty \alpha_j |(\boldsymbol{g}_\alpha^\star)_j|^p \le \frac{\varepsilon}{5} \ . \tag{4.8}$$

As $1 \le p \le 2$, it then follows immediately

$$\sum_{N+1}^\infty |(\boldsymbol{g}_\alpha^\star)_j|^2 \le \frac{\varepsilon}{5} \ . \tag{4.9}$$

Choosing now the iteration index $l$ large enough s.t.

$$\begin{aligned} \sum_{j=1}^\infty \alpha_j |(\boldsymbol{g}_{k_l})_j|^p &= \sum_{j=1}^\infty \alpha_j |(\boldsymbol{g}_\alpha^\star)_j|^p + \tilde{\varepsilon} && (4.10)\\ |(\boldsymbol{g}_{k_l})_j|^{p'} &= |(\boldsymbol{g}_\alpha^\star)_j|^{p'} + \frac{\tilde{\varepsilon}}{N\alpha_j} \qquad |\tilde{\varepsilon}| \le \frac{\varepsilon}{5},\ j = 1, \cdots, N, p' \in \{2, p\} \ . && (4.11) \end{aligned}$$

This is possible for (4.10) because of Lemma 17, and (4.11) can be fulfilled as $N$ is already fixed and $(\boldsymbol{g}_{k_l})_j \to (\boldsymbol{g}^\star_\alpha)_j$ for $l \to \infty$. It follows

$$
\begin{aligned}
\sum_{j=N+1}^{\infty} |(\boldsymbol{g}_{k_l})_j|^2 \quad &\leq \quad \sum_{j=N+1}^{\infty} \alpha_j |(\boldsymbol{g}_{k_l})_j|^p \\
&= \quad \sum_{j=1}^{\infty} \alpha_j |(\boldsymbol{g}_{k_l})_j|^p - \sum_{j=1}^{N} \alpha_j |(\boldsymbol{g}_{k_l})_j|^p \\
&\overset{(4.10)(4.11)}{\leq} \quad \sum_{j=1}^{\infty} \alpha_j |(\boldsymbol{g}^\star_\alpha)_j|^p + |\tilde{\varepsilon}| - \sum_{1}^{N} \alpha_j |(\boldsymbol{g}^\star_\alpha)_j|^p + N \frac{|\tilde{\varepsilon}|}{N} \\
&= \quad \sum_{N+1}^{\infty} \alpha_j |(\boldsymbol{g}^\star_\alpha)_j|^p + 2|\tilde{\varepsilon}| \\
&\overset{(4.9)}{\leq} \quad \frac{3}{5}\varepsilon \ .
\end{aligned} \tag{4.12}
$$

Moreover, since all $\alpha_j \geq 1$, we have by (4.11)

$$
\left| \sum_{j}^{N} |(\boldsymbol{g}_{k_l})_j|^2 - |(\boldsymbol{g}^\star_\alpha)_j|^2 \right| \leq \sum_{j}^{N} \frac{|\tilde{\varepsilon}|}{N} \leq \frac{\varepsilon}{5} \ . \tag{4.13}
$$

Combing estimates (4.9), (4.12) and (4.13) into (4.7), we obtain

$$
D \leq \varepsilon \ ,
$$

and consequently, $\lim_{l\to\infty} \|\boldsymbol{g}_{k_l}\| = \|\boldsymbol{g}^\star_\alpha\|$.

If now for some $l$, $j$ one has $(\boldsymbol{g}_{k_l})_j > 1$, we rescale the sequences and proceed in the same way, i.e. at first we find by (1.4) a scaling factor

$$
\limsup \|\boldsymbol{g}_{k_l}\| \leq \limsup \Psi(\boldsymbol{g}_{k_l}) = \Psi(\boldsymbol{g}^\star_\alpha) =: K \ ,
$$

that rescale the subsequences by $\tilde{\boldsymbol{g}}_{k_l} := \boldsymbol{g}_{k_l}/K$. Hence, we have

$$
\|\tilde{\boldsymbol{g}}_{k_l}\| \leq 1, \qquad\qquad (\tilde{\boldsymbol{g}}_{k_l})_j \to (\tilde{\boldsymbol{g}}^\star_\alpha)_j =: \frac{1}{K}\boldsymbol{g}^\star_\alpha \ .
$$

In particular, one has $|(\tilde{\boldsymbol{g}}_{k_l})_j| \leq 1$ and $\lim_{l\to\infty} \Theta(\tilde{\boldsymbol{g}}_{k_l}) = \Theta(\tilde{\boldsymbol{g}}^\star_\alpha)$. By the same arguments as above we conclude $\lim_{l\to\infty} \|\tilde{\boldsymbol{g}}_{k_l}\| = \|\tilde{\boldsymbol{g}}^\star_\alpha\|$, and thus also $\lim_{l\to\infty} \|\boldsymbol{g}_{k_l}\| = \|\boldsymbol{g}^\star_\alpha\|$. □

In principle, the limits of different convergent subsequences of $\boldsymbol{g}_k$ may differ. Let $\boldsymbol{g}_{k_l} \to \boldsymbol{g}^\star_\alpha$ be a subsequence of $\boldsymbol{g}_k$, and let $\boldsymbol{g}'_{k_l}$ the predecessor of $\boldsymbol{g}_{k_l}$ in $\boldsymbol{g}_k$, i.e. $\boldsymbol{g}_{k_l} = \boldsymbol{g}_i$ and $\tilde{\boldsymbol{g}}'_k = \boldsymbol{g}_{i-1}$. Then we observe, $J^s_\alpha(\boldsymbol{g}_{k_l}, \boldsymbol{g}'_{k_l}) \to J_\alpha(\boldsymbol{g}^\star_\alpha)$. Moreover, as we have $J^s_\alpha(\boldsymbol{g}_{k+1}, \boldsymbol{g}_k) \leq J^s_\alpha(\boldsymbol{g}_k, \boldsymbol{g}_{k-1})$ for all $k$, it turns out that the value of the Tikhonov functional for every limit $\boldsymbol{g}^\star_\alpha$ of a convergent subsequence remains the same, i.e. $J_\alpha(\boldsymbol{g}^\star_\alpha) = const$ .

We may now summarize our findings and give a simple criterion that ensures strong convergence of the whole sequence $\{\boldsymbol{g}_k\}$ towards a critical point of $J_\alpha$.

**Theorem 19** *Assume that there exists at least one isolated limit* $\boldsymbol{g}_\alpha^\star$ *of a subsequence* $\boldsymbol{g}_{k_l}$ *of* $\boldsymbol{g}_k$*. Then* $\boldsymbol{g}_k \to \boldsymbol{g}_\alpha^\star$ *as* $k \to \infty$*. The accumulation point* $\boldsymbol{g}_\alpha^\star$ *is a minimizer for the functional* $J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_\alpha^\star)$ *and fulfills the necessary condition for a minimizer of* $J_\alpha$*.*

*Proof.* As in the proof of Proposition 11 we obtain, $J_\alpha^s(x_\alpha^\star + h, x_\alpha^\star) \geq J_\alpha^s(x_\alpha^\star, x_\alpha^\star) + \frac{C}{2}\|h\|^2$ and with Lemma 4.1 the second assertion is shown. The first assertion can be directly taken from [21]. □

# 5 Regularization properties

A first regularization result can now be stated when restricting the analysis to the very prominent $\ell_1$ case, i.e.

$$\Psi(\boldsymbol{Lg}) = \|\boldsymbol{Lg}\|_{\ell_1} = \sum_{\lambda \in \Lambda} |(\boldsymbol{Lg})_\lambda| \ .$$

The related convex set is then nothing else than

$$\mathcal{C} = \{\boldsymbol{g} \in \ell_2 : \ \sup_{\lambda \in \Lambda} |(\boldsymbol{g})_\lambda| \leq 1\} \ .$$

This yields the componentwise acting projection $P_{\mathcal{C}}(\boldsymbol{Lg}) = \{P_{\mathcal{C}}((\boldsymbol{Lg})_\lambda)\}_{\lambda \in \Lambda}$ with

$$P_{\mathcal{C}}((\boldsymbol{Lg})_\lambda) = \begin{cases} (\boldsymbol{Lg})_\lambda & \text{if } |(\boldsymbol{Lg})_\lambda| \leq 1 \\ \operatorname{sgn}(\boldsymbol{Lg})_\lambda & \text{if } |(\boldsymbol{Lg})_\lambda| > 1 \end{cases} ,$$

and consequently,

$$(I - P_{\mathcal{C}})((\boldsymbol{Lg})_\lambda) = \begin{cases} 0 & \text{if } |(\boldsymbol{Lg})_\lambda| \leq 1 \\ \operatorname{sgn}(\boldsymbol{Lg})_\lambda(|(\boldsymbol{Lg})_\lambda| - 1) & \text{if } |(\boldsymbol{Lg})_\lambda| > 1 \end{cases} .$$

This is the well–known softshrinkage operation with threshold 1, which we denote here by $S_1$. The necessary condition (3.2) thus reads as

$$\boldsymbol{g} = \frac{\alpha}{C}\boldsymbol{L}^* S_1\left(\frac{C}{\alpha}\boldsymbol{L}M(\boldsymbol{g}, \boldsymbol{a})\right) = \boldsymbol{L}^* S_{\frac{\alpha}{C}}\left(\boldsymbol{L}M(\boldsymbol{g}, \boldsymbol{a})\right).$$

For this specific case we may now state the following result.

**Theorem 20** *Let* $y^\delta \in Y$ *with* $\|y^\delta - y\| \leq \delta$ *and let* $\alpha(\delta)$ *be chosen with* $\alpha(\delta) \to 0$ *and* $\delta^2/\alpha(\delta) \to 0$ *as* $\delta \to 0$*. Then every sequence* $\{\boldsymbol{g}_{\alpha_k}^{\delta_k}\}$ *of minimizers of the functional* $J_{\alpha_k}(\boldsymbol{g})$*, defined in* (1.3) *where* $\delta_k \to 0$ *and* $\alpha_k = \alpha(\delta_k)$ *has a convergent subsequence. The limit of every convergent subsequence is a solution of* $T(F^*\boldsymbol{g}) = y$ *with minimal value of* $\Psi(\boldsymbol{Lg})$*. If, in additition, the solution* $\boldsymbol{g}^\dagger$ *with minimal* $\Psi(\boldsymbol{Lg})$ *is unique, then we have*

$$\lim_{\delta \to 0} \boldsymbol{g}_{\alpha(\delta)}^\delta = \boldsymbol{g}^\dagger \ . \tag{5.1}$$

*Proof.* Let $\alpha_k$ and $\delta_k$ be as above , and $\boldsymbol{g}^\dagger$ a solution of $T(F^*\boldsymbol{g})$ with minimal value of $\Psi(\boldsymbol{L}\boldsymbol{g})$. As $\boldsymbol{g}_{\alpha_k}^{\delta_k}$ is a minimizer of $J_{\alpha_k}$, we have

$$\|T(F^*\boldsymbol{g}_{\alpha_k}^{\delta_k}) - y^{\delta_k}\|^2 + 2\alpha_k\Psi(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k}) \le \delta_k^2 + 2\alpha_k\Psi(\boldsymbol{L}\boldsymbol{g}^\dagger) \ . \tag{5.2}$$

Hence we have $\|T(F^*\boldsymbol{g}_{\alpha_k}^{\delta_k}) - y^{\delta_k}\|^2 \le \delta_k^2 + 2\alpha_k\Psi(\boldsymbol{L}\boldsymbol{g}^\dagger)$ and thus

$$\lim_{k\to\infty} T(F^*\boldsymbol{g}_{\alpha_k}^{\delta_k}) = y \ . \tag{5.3}$$

Moreover, we have $\Psi(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k}) \le \delta_k^2/\alpha_k(\delta_k) + \Psi(\boldsymbol{L}\boldsymbol{g}^\dagger)$, which yields

$$\limsup_k \|\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k}\|_{\ell_2} \overset{(1.4)}{\le} \limsup_k \Psi(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k}) \le \Psi(\boldsymbol{L}\boldsymbol{g}^\dagger) \ , \tag{5.4}$$

i.e. $\|\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k}\|_{\ell_2}$ and $\|\boldsymbol{g}_{\alpha_k}^{\delta_k}\|_{\ell_2}$ are bounded, and the sequence has a weakly convergent subsequence, again denoted by $\{\boldsymbol{g}_{\alpha_k}^{\delta_k}\}$,

$$\boldsymbol{g}_{\alpha_k}^{\delta_k} \rightharpoonup \boldsymbol{g}^\star \ . \tag{5.5}$$

In particular, as $T$ is strongly continuous,

$$y \overset{(5.3)}{=} \lim_{k\to\infty} T(F^*\boldsymbol{g}_{\alpha_k}^{\delta_k}) = T(F^*\boldsymbol{g}^\star) \ ,$$

and thus $\boldsymbol{g}^\star$ is a solution of $T(F^*\boldsymbol{g}) = y$. By assumption, $\Psi$ is weak semi-continuous, and thus we derive

$$\Psi(\boldsymbol{L}\boldsymbol{g}^\star) \le \limsup_k \Psi(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k}) \overset{(5.4)}{\le} \Psi(\boldsymbol{L}\boldsymbol{g}^\dagger) \le \Psi(\boldsymbol{L}\boldsymbol{g}^\star) \ . \tag{5.6}$$

The last inequality follows from the fact that $\boldsymbol{g}^\dagger$ is a solution with minimal value of $\Psi(\boldsymbol{L}\cdot)$. As a consequence, $\Psi(\boldsymbol{L}\boldsymbol{g}^\star) = \Psi(\boldsymbol{L}\boldsymbol{g}^\dagger)$, and $\boldsymbol{g}^\star$ is also a solution with minimal $\Psi$-value.
Next, we need to rewrite the absolute value of a real number. Defining

$$\varphi(x,h) = \begin{cases} -\mathrm{sgn}(x)\cdot h & \text{if } 0 \ne \mathrm{sgn}(x) = \mathrm{sgn}(h) \text{ and } |x| > |h| \\ (\mathrm{sgn}(x)\cdot h - 2|x|) & \text{if } 0 \ne \mathrm{sgn}(x) = \mathrm{sgn}(h) \text{ and } |x| \le |h| \\ |h| & \text{if } 0 \ne \mathrm{sgn}(x) = -\mathrm{sgn}(h) \\ |h| & \text{if } x = 0 \ , \end{cases} \tag{5.7}$$

we obtain

$$|x - h| = |x| + \varphi(x,h) \ . \tag{5.8}$$

Setting $x = (\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j$, $h = (\boldsymbol{L}\boldsymbol{g}^\star)_j$ yields

$$\begin{aligned} \Psi(\boldsymbol{L}(\boldsymbol{g}_{\alpha_k}^{\delta_k} - \boldsymbol{g}^\star)) &= \sum_j |(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j - (\boldsymbol{L}\boldsymbol{g}^\star)_j| \\ &= \sum_j |(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j| + \sum_j (\varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j) \\ &= \Psi(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k}) + \sum_j (\varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j) \\ &\overset{(5.6)}{\le} \Psi(\boldsymbol{L}\boldsymbol{g}^\star) + \sum_j (\varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j) \quad . \end{aligned}$$

By the definition of $\varphi(x,h)$ in (5.7) follows

$$|\varphi(x,h)| = \begin{cases} |-\operatorname{sgn}(x)\cdot h| \le |h| & \text{if } 0 \ne \operatorname{sgn}(x) = \operatorname{sgn}(h) \text{ and } |x| > |h| \\ |\operatorname{sgn}(x)\cdot h - 2|x|| \le 3|h| & \text{if } 0 \ne \operatorname{sgn}(x) = \operatorname{sgn}(h) \text{ and } |x| \le |h| \\ |h| & \text{if } 0 \ne \operatorname{sgn}(x) = -\operatorname{sgn}(h) \\ |h| & \text{if } x = 0 \ , \end{cases} \tag{5.9}$$

i.e.

$$|\varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j)| \le 3|(\boldsymbol{L}\boldsymbol{g}^\star)_j|$$

and thus

$$\sum_j (\varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j) \le 3 \sum_j |(\boldsymbol{L}\boldsymbol{g}^\star)_j)| = 3\Psi(\boldsymbol{L}\boldsymbol{g}^\star) \ ,$$

i.e. $\sum_j 3|(\boldsymbol{L}\boldsymbol{g}^\star)_j)|$ dominates $\sum_j(\varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j)$, and we can interchange limit and sum,

$$\lim_{k\to\infty} \sum_j (\varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j) = \sum_j \lim_{k\to\infty} (\varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j) \tag{5.10}$$

As $\boldsymbol{g}_{\alpha_k}^{\delta_k} \xrightarrow{\ell_2} \boldsymbol{g}^\star$, we have in particular $(\boldsymbol{g}_{\alpha_k}^{\delta_k})_j \to (\boldsymbol{g}^\star)_j$ for $k \to \infty$, and thus $(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j \to (\boldsymbol{L}\boldsymbol{g}^\star)_j$ for $k \to \infty$. Now assume $(\boldsymbol{L}\boldsymbol{g}^\star)_j \ne 0$ for some $j$. Then there exists $k_0$ s.t. $(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j \ne 0$ and $\operatorname{sgn}((\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j) = \operatorname{sgn}((\boldsymbol{L}\boldsymbol{g}^\star)_j)$ for all $k \ge k_0$. According to the definition (5.7) of $\varphi$, we have thus for $k \ge k_0$

$$\varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j) = \begin{cases} -\operatorname{sgn}((\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j)\cdot(\boldsymbol{L}\boldsymbol{g}^\star)_j = -|(\boldsymbol{L}\boldsymbol{g}^\star)_j| & \text{for } |(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j| > |(\boldsymbol{L}\boldsymbol{g}^\star)_j| \\ \operatorname{sgn}((\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j)\cdot(\boldsymbol{L}\boldsymbol{g}^\star)_j) - 2|(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j| = |(\boldsymbol{L}\boldsymbol{g}^\star)_j| - 2|(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j| & \\ & \text{for } |(\boldsymbol{L}\boldsymbol{g}_{\alpha_k}^{\delta_k})_j| > |(\boldsymbol{L}\boldsymbol{g}^\star)_j| \end{cases}$$

and thus

$$\lim_{k\to\infty} \varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j) = -|(\boldsymbol{L}\boldsymbol{g}^\star)_j| \ .$$

Consequently,

$$\begin{aligned} 0 &\le \lim_{k\to\infty} \Psi(\boldsymbol{L}(\boldsymbol{g}_{\alpha_k}^{\delta_k} - \boldsymbol{g}^\star)) \le \Psi(\boldsymbol{L}\boldsymbol{g}^\star) + \lim_{k\to\infty} \sum_j (\varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j) \\ &= \Psi(\boldsymbol{L}\boldsymbol{g}^\star) + \sum_j \lim_{k\to\infty} (\varphi((\boldsymbol{g}_{\alpha_k}^{\delta_k})_j, (\boldsymbol{g}^\star)_j) = \Psi(\boldsymbol{L}\boldsymbol{g}^\star) - \sum_j |(\boldsymbol{L}\boldsymbol{g}^\star)_j| = 0 \ , \end{aligned}$$

which proves $\boldsymbol{g}_{\alpha_k}^{\delta_k} \to \boldsymbol{g}^\star$ with respect to $\Psi$ and, because of (1.4), also with respect to $\ell_2$. If $\boldsymbol{g}^\star$ is unique, our assertion about the convergence of $\boldsymbol{g}_{\alpha(\delta)}^{\delta}$ follows by the convergence principles from the fact that every sequence has a convergent subsequence with the same limit $\boldsymbol{g}^\dagger$.

$\square$

We wish to remark that uniqueness can only be expected in the basis setting. In a frame lore, every function has several representations with respect to the given frame, and thus the minimizer cannot be unique.

*We finally summarize our proposed scheme:* Assume that all the conditions we have imposed in the previous sections apply to our problem and, moreover, assume we have a parameter rule at hand that fulfills the conditions of Theorem 20. Then the regularization algorithm (at least for the $\ell_1$ case) goes as follows:

- For given error level $\delta$, pick a regularization parameter according to the conditions of Theorem 20, and choose $\boldsymbol{g}_0$
- pick an admissible $C$
- $[\boldsymbol{g}_\alpha^\star]$ = Iteration($T$, $y^\delta$, C, $\alpha$, $\boldsymbol{g}_0$):
  $\boldsymbol{g}_{k+1} = \arg\min_{\boldsymbol{g}} J_\alpha^s(\boldsymbol{g}, \boldsymbol{g}_k)$ (solved by a projected fixed point iteration)
  $\boldsymbol{g}_\alpha^\star = \lim_{k\to\infty} \boldsymbol{g}_k$
  **end**

In practice (treatment of limits), we have to incorporate stopping rules that will slightly modify this scheme:

- For given error level $\delta$, pick a regularization parameter according to the conditions of Theorem 20, and choose $\boldsymbol{g}_0$
- choose two tolerances $\tau_1, \tau_2$
- pick an admissible $C$
- $[\boldsymbol{g}_\alpha^\star]$ = Iteration($T$, $y^\delta$, C, $\alpha$, $\tau_1$, $\tau_2$)
  $k = 0$
  **while** $\|\boldsymbol{g}_{k+1} - \boldsymbol{g}_k\|_{\ell_2} > \tau_1$
  $\quad l = 0,\ \boldsymbol{g}_{k,0} = \boldsymbol{g}_k$
  $\quad$ **while** $\|\boldsymbol{g}_{k,l} - \boldsymbol{g}_{k,l+1}\|_{\ell_2} > \tau_2$
  $\qquad l = l + 1$
  $\qquad \boldsymbol{g}_{k,l} = \Phi_{\alpha,\mathcal{C}}(\boldsymbol{g}_{k,l-1}, \boldsymbol{g}_k)$
  $\qquad$ **end**
  $\quad \boldsymbol{g}_{k+1} = \boldsymbol{g}_{k,l}$
  $\quad k = k + 1$
  **end**
- $\boldsymbol{g}_\alpha^\star = \boldsymbol{g}_k$

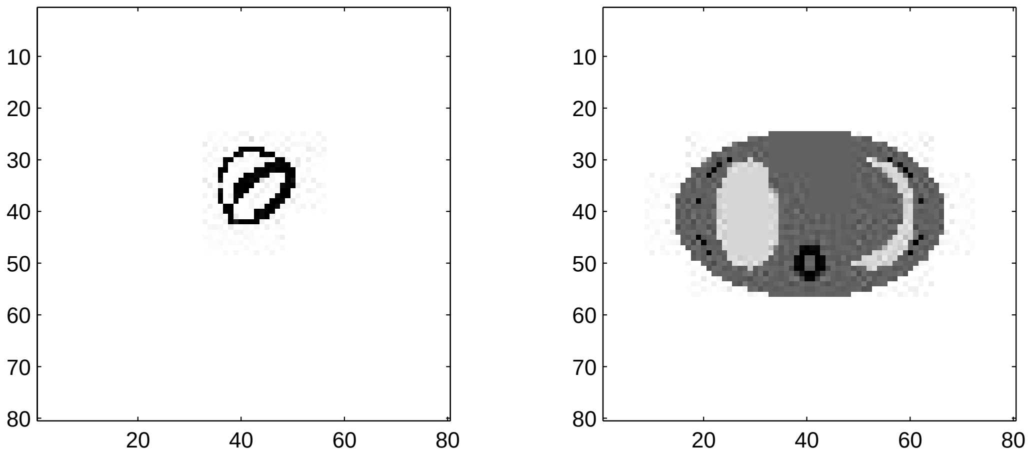



Figure 1: Activity function $f_*$ (left) and attenuation function $\mu_*$ (right). The activity function models a cut through the heart.

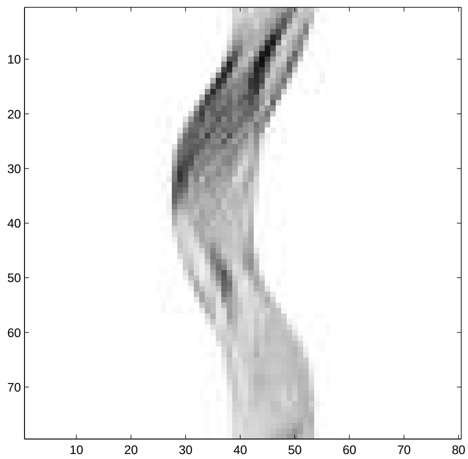



Figure 2: Generated data $g(s, \omega) = R(f_*, \mu_*)(s, \omega)$.

# 6 Numerical Illustration

In this section, we want to present some first numerical results of a sparse reconstruction from SPECT (Single Photon Emission Computed Tomography). SPECT is a medical imaging technique where one aims to reconstruct a radioactivity distribution $f$ from radiation measurements outside the body. The measurements are described by the attenuated Radon transform (ATRT)

$$y = R(f, \mu)(s, \omega) = \int_{\mathbb{R}} f(s\omega^\perp + t\omega) e^{-\int_t^\infty \mu(s\omega^\perp + r\omega) dr} dt \ . \tag{6.1}$$

As the measurements depend on the (usually also unknown) density distribution $\mu$ of the tissue, we have to solve a nonlinear problem in $(f, \mu)$. An throughout analysis of the nonlinear ATRT was presented by Dicken [11], and several approaches for its solution were proposed in

[2, 12, 24, 25, 20, 17, 18, 19]. If the ATRT operator is considered with

$$D(R) = H_0^{s_1}(\Omega) \times H_0^{s_2}(\Omega) ,$$

where $H_0^s(\Omega)$ denotes a Sobolev space over a bounded area $\Omega$ with zero boundary conditions and smoothness $s$, then the operator is twice continuous Frèchet differentiable with Lipschitz continuous first derivative, if $s_1,\ s_2$ are chosen large enough. A possible choice for these parameters that also reflects the smoothness properties of activity and density distribution is $s_1 > 4/9$ and $s_2 = 1/3$. For more details we refer to [18, 10]. Additionally, it has been shown that conditions (1.6), (1.8) hold [16]. For our test computations, we will use the so called MCAT – phantom [23], see Figure 1. Both functions were given as $80 \times 80$ pixel images. The sinogram data was gathered on 79 angles, equally spaced over 360 degree, and 80 samples. The sinogram belonging to the MCAT phantom is shown in Figure 2. At first, we have to choose the underlaying frame or basis on which we put the sparsity constraint. Since a wavelet expansion might sparsely represent images/functions (better than pixel basis), we have chosen a wavelet basis (here Daubechies wavelets of order two) to represent $(f, \mu)$, i.e.

$$(f,\mu) = \left( \sum_k c(f)_k \phi_{0,k} + \sum_{j\geq 0,i,k} d(f)^i_{j,k} \psi^i_{j,k} \ , \ \sum_k c(\mu)_k \phi_{0,k} + \sum_{j\geq 0,i,k} d(\mu)^i_{j,k} \psi^i_{j,k} \right) .$$

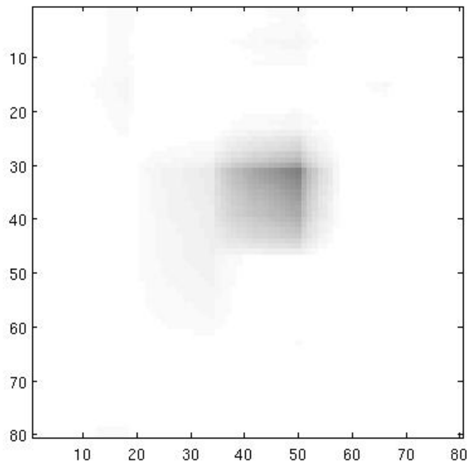
For more details we refer the reader to [6]. Moreover, for our implementation we have chosen $\boldsymbol{L} = I$, i.e. the penalty is given by $\Psi(\cdot) = \| \cdot \|_{\ell_1}$. Our algorithm requires to pick values $\tau_1,\ \tau_2$ for the termination of the inner and outer iteration. In our implementation, the inner iteration was stopped if the *relative* error was smaller than $10^{-6}$, i.e.

$$\frac{\left(\|f_{k,l} - f_{k,l+1}\|^2 + \|\mu_{k,l} - \mu_{k,l+1}\|^2\right)^{1/2}}{\left(\|f_{k,l}\|^2 + \|\mu_{k,l}\|^2\right)^{1/2}} \leq 10^{-6} .$$

For the outer iteration, a relative error of $10^{-5}$ was used. The convergence speed of the iteration depends heavily on the choice of the constant $C$ in (1.5). According to our convergence analysis, it has to be chosen reasonably large. However, a large $C$ speeds up the convergence of the inner iteration, but decreases the speed of convergence of the outer iteration. In our example, we needed only 2-4 inner iteration, but the outer iteration required about 5000 iterations. As the minimiztion in the quadratic case needed much less iterations, this suggests that the speed of convergence also increases with $p$.

According to (1.4), the functional $\Psi$ will always have a bigger value than $\| \cdot \|_{\ell_2}$. If $\Psi(\boldsymbol{g})$ is not to large, then it will also dominate $\|\boldsymbol{g}\|^2_{\ell_2}$, which also represents the classical $L_2-$norm, and we might conclude that reconstructions with the classical quadratic Hilbert space constraint and sparsity constraint will not give comparable results if the same regularization parameter is used. As $\Psi$ is dominant, we expect a smaller (optimal) regularization parameter in the case of the penalty term $\Psi$. This is confirmed by our first test computations: Figure 3 shows the reconstructions from noisy data where the regularization parameter was chosen as $\alpha = 350$. The reconstruction with the quadratic Hilbert space penalty (we have used the $L_2$ norm) is already quite good, whereas the reconstruction for the sparsity constraint is still far off. In fact, if we consider Morozov's discrepancy principle, then the regularization parameter in the quadratic case has been chosen optimal, as we observe

$$\|y^\delta - A(f^\delta_\alpha, \mu^\delta_\alpha)\| \approx 2\delta .$$



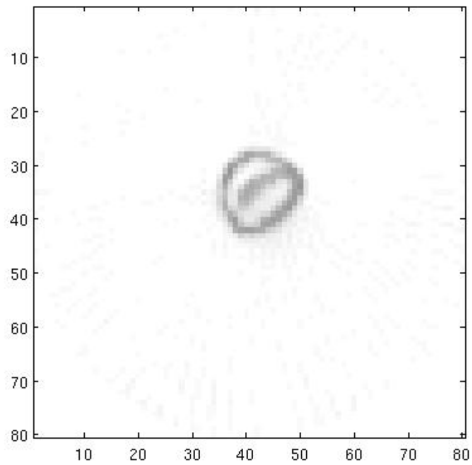




Figure 3: Reconstructions with 5% noise and $\alpha = 350$: sparsity constraint (left) and Hilbert space constraint (right).

To obtain a reasonable basis for comparison, we adjusted the regularization parameter $\alpha$ such that the residual had also a magnitude of $2\delta$ in the sparsity case, which occurred for $\alpha = 5$ . The reconstruction can be seen in Figure 4

A visual inspection shows that the reconstruction with sparsity constraint yields much sharper contours. In particular, the absolute values of $f$ in the heart are higher in the sparsity case, and the artefacts are not as bad as in the quadratic constraint case, as can be seen in Figure 5. It shows a plot of the values of the activity function for both reconstructions along a row in the image in Figures 3 and 4 respectively. The left graph shows the values at a line that goes through the heart, and right graph shows the values along a line well outside the heart, where only artefacts occur. Clearly, both reconstructions are different, but it certainly needs much more computations in order to decide in which situations a sparsity constraint has to be preferred. A histogram plot of the wavelet coefficients for both reconstructions shows that the reconstruction with sparsity constraint has much more small coefficients - it is, as we did expect, a sparse reconstruction, see Figure 6.

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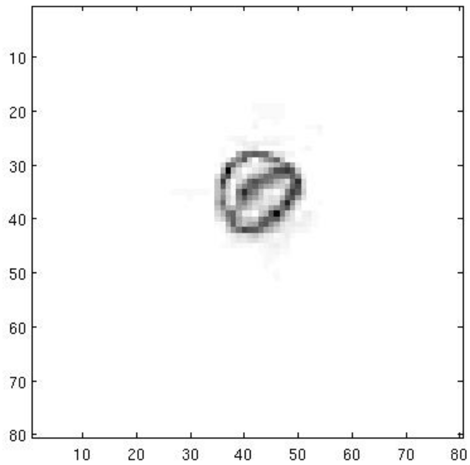
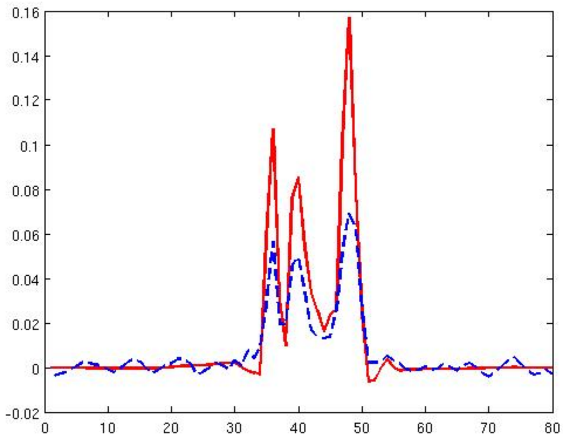



Figure 4: Reconstruction with sparsity constraint and 5% noise. The regularization parameter ($\alpha = 5$) was chosen such that $\|y^\delta - A(f_\alpha^\delta, \mu_\alpha^\delta)\| \approx 2\delta$



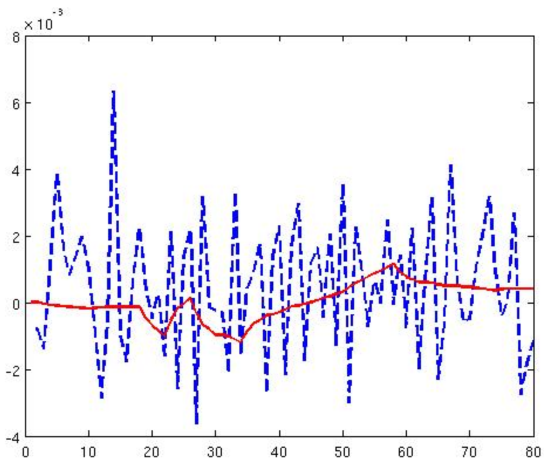




Figure 5: Values of the reconstructed activity function through the heart (left) and well below the heart (right). Solid line: reconstruction with sparsity constraint, dashed line: quadratic Hilbert space penalty

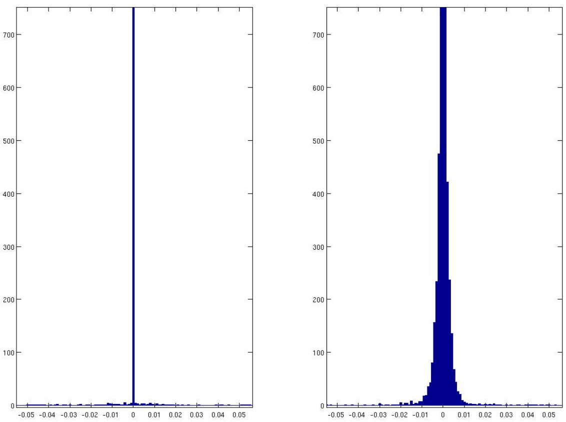


Figure 6: Histogram plot of the wavelet coefficient of the reconstructions. Left: sparsity constraint, Right: quadratic Hilbert space constraint.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