Tikhonov Replacement Functionals for Iteratively Solving Nonlinear Operator Equations*

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Abstract

We shall be concerned with the construction of Tikhonov–based iteration schemes for solving nonlinear operator equations. In particular, we are interested in algorithms for the computation of a minimizer of the Tikhonov functional. To this end, we introduce a replacement functional, that has much better properties than the classical Tikhonov functional with nonlinear operator. Namely, the replacement functional is globally convex and can be effectively minimized by a fixed point iteration. Based on the minimizers of the replacement functional, we introduce an iterative algorithm that converges towards a critical point of the Tikhonov functional, and under additional assumptions to the nonlinear operator F, to a global minimizer. Combining our iterative strategy with an appropriate parameter selection rule, we obtain convergence and convergence rates. The performance of the resulting numerical scheme is demonstrated by solving the nonlinear inverse SPECT (Single Photon Emission Computerized Tomography) problem.

1 Introduction

In this paper, we consider the computation of an approximation to a solution of a nonlinear operator equation

$$F(x) = y \ , \tag{1.1}$$

where $F : X \to Y$ is an ill-posed operator between Hilbert spaces X, Y. If only noisy data y^δ with

$$\|y^\delta - y\| \leq \delta \tag{1.2}$$

are available, problem (1.1) has to be stabilized by regularization methods. In recent years, many of the well known methods for linear ill-posed problems have been generalized to nonlinear operator equations. Unfortunately, it turns out that convergence and convergence rates can be shown only under severe restrictions to the operator for most methods. For example, convergence for Landweber method can be shown only if the operator fulfills

$$\|F(x) - F(\tilde{x}) - F'(\tilde{x})(x - \tilde{x})\| \leq \eta \|x - \tilde{x}\| \qquad \text{with} \ \ \eta < 1/2 \ , \tag{1.3}$$

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whereas convergence rates are only available if, for a solution $x^\dagger$ of (1.1), there exists a family of bounded operators R_x with

$$F'(x) = R_x F'(x^\dagger) \;\text{ and }\; \|I - R_x\| \leq K\|x - x^\dagger\| \;.$$

For other prominent iterative methods like Gauss–Newton [1, 2], Levenberg–Marquardt [9], conjugate gradient [10] and Newton–like methods [13, 6], convergence can be shown under similar restrictions as (1.3). To obtain convergence rates, much stronger restrictions have to be assumed.

An alternative to the above mentioned iterative methods is Tikhonov regularization, where an approximation to the solution of (1.1) is obtained by minimizing the Tikhonov functional $J_\alpha(x)$,

$$J_\alpha(x) \;=\; \|y^\delta - F(x)\|^2 + \alpha\|x - \bar{x}\|^2 \;, \tag{1.4}$$

$$x_\alpha^\delta \;=\; \arg\min_x J_\alpha(x) \;. \tag{1.5}$$

The advantage of Tikhonov regularization is that convergence of the method, i.e. $x_\alpha^\delta \to x^\dagger$ for $\delta \to 0$ and an appropriate parameter choice $\alpha = \alpha(\delta)$ holds under weak assumptions to the operator, see, e.g., [8], and convergence rates are obtained for Fréchet differentiable operators with Lipschitz continuous derivative. However, the difficulties for Tikhonov regularization are a proper choice of the regularization parameter [22, 17] and the computation of the minimizer of the Tikhonov functional. As the functional is no longer convex for nonlinear operators F, J_α can even have local minimizers, and classical optimization routines might fail. Recently, we have introduced iterative methods for the minimization of the Tikhonov functional that reconstruct a global minimizer of the Tikhonov functional provided a smoothness assumption $x^\dagger - \bar{x} = F'(x^\dagger)^*\omega$ with small $\|\omega\|$ holds. We wish to remark that it might be difficult to show such smoothness conditions for practical problems, and for exponentially ill-posed problems Hölder-type smoothness conditions will not hold, see [12]. Thus it would be advantageous to construct iterative methods that reconstruct a minimizer of the Tikhonov functional under different assumptions. But this seems to remain a pipe dream: even here in this paper we had to incorporate some smoothness conditions to prove global minimizing properties of the reconstructed solution. However, all the here made assumptions on F are within the frame of nonlinear technologies and they are not that strong than for most of the above quoted iterative schemes.

In this paper, we will investigate a method that *always* finds a *critical point* of the Tikhonov functional. Under additional assumptions on the operator and a smoothness condition on the solution we can then assure that this critical point is a *global minimizer* of J_α.

The basic idea for our new iteration scheme goes as follows: consider the Tikhonov variational formulation of the inverse problem. Due to the nonlinearity, a direct reconstruction of the global minimizer is not possible. Thats why we aim to solve instead of the pure Tikhonov functional a sequence of so-called surrogate or replacement functionals. This idea is borrowed from linear regularization methods with general and mixed smoothness constraints, see e.g. [4, 5]. The intention in [4, 5] is to decouple the variational equations with respect to the basis coefficients of the solution caused by the linear operator. The cost of dealing with a decoupled system of equations is an iteration process from which strong convergence properties can be shown. The situation in the nonlinear case is completely different and due to the impact of the Fréchet derivative one cannot expect to end up with similar schemes than in [4, 5]. However, the basic advantage of using replacement functionals is that each of the functionals is under certain conditions on the construction process globally convex. The minimization results then in an easy fixed point iteration. Defining now an iteration process by iteratively minimizing a sequence of replacement functionals, we can show that the sequence of minimizing elements of each individual fixed point iteration converges in norm towards a critical point of the Tikhonov functional of the nonlinear inverse problem. Imposing additional assumptions (on the quadratic remainder of the Taylor series expression of our operator under consideration, and a smoothness condition) we obtain a uniqueness result, i.e. we are able to show that the reconstructed critical point is a global minimizer. Finally, applying a proper parameter choice rule, we are able to adopt classical convergence/order optimality results for Tikhonov regularization methods.

The remaining paper is organized as follows: In Section 2, we state the scope of the problem. In Section 3, we explain how the replacement functionals are constructed and we minimize them in Section 4. The main result of the paper is presented in Section 5: strong convergence of the iterates towards a global minimizer. We end this paper with Section 6 in which we demonstrate the capabilities of the proposed scheme by solving the nonlinear SPECT problem.

2 The scope of the problem

We consider the problem of deriving a minimizer of the Tikhonov functional

$$J_\alpha(x) = \|y^\delta - F(x)\|^2 + \alpha\|x - \bar{x}\|^2 \ . \tag{2.1}$$

Due to the nonlinearity of the operator F, the minimizer of the functional might not be unique, or there might exist even local minimizers, such that a standard minimizing algorithm can fail in reconstructing a global minimizer. In order to obtain an easier problem which hopefully has a unique solution, we replace the functional J_α by

$$\tilde{J}_\alpha(x, a) := \|y^\delta - F(x)\|^2 + \alpha\|x - \bar{x}\|^2 + C\|x - a\|^2 - \|F(x) - F(a)\|^2 \tag{2.2}$$

and proceed as follows:

1. Pick x_0 and some proper constant $C > 0$
2. Derive a sequence $\{x_k\}_{k=0,1,\dots}$ by the iteration:

$$x_{k+1} = \arg\min_x \tilde{J}_\alpha(x, x_k) \qquad k = 0, 1, 2, \dots$$

The overall goal of this paper is to show that the sequence $\{x_k\}_{k=0,1,\dots}$ converges in norm topology towards a global minimizer of the Tikhonov functional (2.1).

In order to achieve this result we proceed in two steps: First, we aim to show norm convergence of the iterates x_k towards a critical point of the Tikhonov functional. In a second step, we verify that the reconstructed critical point is equal to a global minimizer of the Tikhonov functional. To make this program running, we have to restrict ourselves as follows:

- For the first step we limit the analysis to nonlinear operators F for which

$$x_k \xrightarrow{w} x \Longrightarrow F(x_k) \to F(x) \text{ and } F'(x_k)^* z \to F'(x)^* z \text{ for all } z \; , \tag{2.3}$$

$$\|F'(x) - F'(\tilde{x})\| \leq L\|x - \tilde{x}\| \; . \tag{2.4}$$

  It may happen that F already meets these conditions as an operator from $X \to Y$. If not, this can be achieved by assuming more regularity of the solution, i.e. we have to change the domain of F a little. To this end, let us assume that there exists a function space X^s, and a compact embedding operator $i^s : X^s \to X$. Now we can consider

$$\tilde{F} = F \circ i^s : X^s \longrightarrow Y \; .$$

  We obtain

$$\|\tilde{F}'(x) - \tilde{F}'(\tilde{x})\| \leq L\|x - \tilde{x}\|_X \leq L\|x - \tilde{x}\|_{X^s} \; . \tag{2.5}$$

  If now $x_k \xrightarrow{w} x$ in X^s, then $x_k \to x$ in X and, moreover, (2.5) yields $\tilde{F}'(x_k) \to \tilde{F}'(x)$ and $\tilde{F}'(x_k)^* \to \tilde{F}'(x)^*$ in the operator norm. This argument applies to arbitrary nonlinear continuous and Fréchet differentiable operators $F : X \to Y$ with continuous Lipschitz derivative as long as a function space X^s with compact embedding i^s to X is available.

- To process the second step, we additionally impose that $x^\dagger$ fulfills a smoothness condition, F is twice differentiable, and that

$$\|F(x) - F(\tilde{x}) - F'(\tilde{x})(x - \tilde{x})\| \leq \|F(x) - F(\tilde{x})\| \; , \tag{2.6}$$

  which is a condition on the quadratic remainder of the Taylor series expansion of F.

3 On the proper definition of the replacement functional

By the definition of $\tilde{J}_\alpha$ in (2.2) it is not clear whether the functional is positive definite or even bounded from below. This will be clarified in this section, i.e. we will show that this is the case provided the constant C is chosen properly.

For given $\alpha > 0$ and x_0 we define a ball $K_r(\bar{x})$ with radius r around $\bar{x}$, where the radius is given by

$$r^2 := \begin{cases} \frac{\|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2}{\alpha} & \text{for } \alpha < 1 \\ \\ \|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2 & \text{for } \alpha \geq 1 \end{cases} \; . \tag{3.1}$$

This obviously ensures, $x_0 \in K_r(\bar{x})$. Furthermore, we define the constant C by

$$C := \max\left\{4,\; 2\left(\sup_{x \in K_r(\bar{x})} \|F'(x)\|\right)^2, 2L\sqrt{\|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2}\right\} \quad , \tag{3.2}$$

where L is the Lipschitz constant of the Fréchet derivative of F. We assume that x_0 was chosen such that $r < \infty$ and $C < \infty$.

Lemma 1 *Let r and C be chosen by (3.1), (3.2). Then*

$$C\|x - x_0\|^2 - \|F(x) - F(x_0)\|^2 \geq 0 \tag{3.3}$$

for all $x \in K_r(\bar{x})$, and, thus, $J_\alpha(x) \leq \tilde{J}_\alpha(x, x_0)$.

Proof. By Taylors expansion we have

$$F(x+h) = F(x) + \int_0^1 F'(x+\tau h)h\, d\tau$$

and thus

$$\|F(x) - F(x+h)\| \leq \int_0^1 \|F'(x+\tau h)\|\|h\| d\tau \leq \sup_{x \in K_r(\bar{x})} \|F'(x)\|\|h\| \; .$$

Consequently, we get for all $x \in K_r(\bar{x})$

$$\begin{aligned} C\|x - x_0\|^2 - \|F(x) - F(x_0)\|^2 &\geq C\|x - x_0\|^2 - (\sup_{x \in K_r(\bar{x})} \|F'(x)\|)^2 \|x - x_0\|^2 \\ &= \frac{C}{2}\|x - x_0\|^2 \geq 0 \; , \end{aligned}$$

and the functional $\tilde{J}_\alpha(x, x_0)$ is positive for all $x \in K_r(\bar{x})$. □

Next, we show that this carries over to all of the iterates:

Proposition 2 *Let x_0, α be given and r, C be defined by (3.1), (3.2). Then the functionals $\tilde{J}_\alpha(x, x_k)$ are bounded from below for all $k \in \mathbb{N}$ and have thus minimizers. For the minimizer x_{k+1} of $\tilde{J}_\alpha(x, x_k)$ holds $x_{k+1} \in K_r(\bar{x})$.*

Proof. The proof will be done by induction. For $k = 1$, we show in a first step that $\tilde{J}_\alpha(x, x_0)$ is bounded from below. We have

$$\|y^\delta - F(x)\|^2 = \|y^\delta - F(x_0)\|^2 + \|F(x_0) - F(x)\|^2 + 2\langle y^\delta - F(x_0), F(x_0) - F(x)\rangle \; . \tag{3.4}$$

Thus,

$$\begin{aligned} \tilde{J}_\alpha(x, x_0) - \alpha\|x - \bar{x}\|^2 &= \|y^\delta - F(x_0)\|^2 + 2\langle y^\delta - F(x_0), F(x_0) - F(x)\rangle + C\|x - x_0\|^2 \\ &\geq \|y^\delta - F(x_0)\|^2 - 2\|y^\delta - F(x_0)\|\|F(x_0) - F(x)\| + C\|x - x_0\|^2 \; . \end{aligned} \tag{3.5}$$

Again by Taylor expansion, we get

$$\|F(x_0) - F(x)\| \leq \|F'(x_0)\|\|x_0 - x\| + \frac{L}{2}\|x_0 - x\|^2 \; . \tag{3.6}$$

Now let us assume that $\tilde{J}_\alpha(x, x_0)$ is not bounded from below. As F is continuous, there exists a sequence $\{x_l\}_{l\in\mathbb{N}}$ with $\|x_l\| \to \infty$ and $\tilde{J}_\alpha(x_l, x_0) \to -\infty$. In particular, for l large enough, follows from (3.6)

$$\|F(x_0) - F(x_l)\| \leq L\|x_0 - x_l\|^2 ,$$

and combining this estimate with (3.5) yields

$$\tilde{J}_\alpha(x_l, x_0) - \alpha\|x_l - \bar{x}\|^2 \geq \|y^\delta - F(x_0)\|^2 - 2L\|y^\delta - F(x_0)\|\|x_l - x_0\|^2 + C\|x_l - x_0\|^2 .$$

From the definition of C in (3.2) follows $2L\|y^\delta - F(x_0)\| \leq C$ and thus

$$\tilde{J}_\alpha(x_l, x_0) - \alpha\|x_l - \bar{x}\|^2 \geq \|y^\delta - F(x_0)\|^2 \geq 0,$$

in contradiction to our assumption $\tilde{J}_\alpha(x_l, x_0) \to -\infty$, and thus $\tilde{J}_\alpha(x, x_0)$ is bounded from below. By the same argument, we find $\tilde{J}_\alpha(x_l, x_0) \geq \alpha\|x_l - \bar{x}\|^2 \to \infty$ for any sequence x_l with $\|x_l\| \to \infty$ and thus the functional is coercive and has a minimizer x_1.

As in (3.5), we get by using (3.6)

$$\begin{aligned}\tilde{J}_\alpha(x_1, x_0) - \alpha\|x_1 - \bar{x}\|^2 &\geq \|y^\delta - F(x_0)\|^2 + 2\langle y^\delta - F(x_0), F(x_0) - F(x_1)\rangle + C\|x_1 - x_0\|^2 \\ &\geq \|y^\delta - F(x_0)\|^2 - 2\|y^\delta - F(x_0)\|\|F'(x_0)\|\|x_1 - x_0\| \\ &\quad -L\|y^\delta - F(x_0)\|\|x_1 - x_0\|^2 + C\|x_1 - x_0\|^2\end{aligned}$$

By (3.2), we have $C/2 \geq L\|y^\delta - F(x_0)\|$, and thus

$$\tilde{J}_\alpha(x_1, x_0) - \alpha\|x_1 - \bar{x}\|^2 \geq \|y^\delta - F(x_0)\|^2 - 2\|y^\delta - F(x_0)\|\|F'(x_0)\|\|x_1 - x_0\| + \frac{C}{2}\|x_1 - x_0\|^2 . \quad (3.7)$$

As $x_0 \in K_r(\bar{x})$, it follows from (3.2) that $\|F'(x_0)\| \leq \sqrt{C/2}$ holds, and we get finally

$$\begin{aligned}\tilde{J}_\alpha(x_1, x_0) - \alpha\|x_1 - \bar{x}\|^2 &\geq \|y^\delta - F(x_0)\|^2 - 2\frac{\sqrt{C}}{\sqrt{2}}\|y^\delta - F(x_0)\|\|x_1 - x_0\| + \frac{C}{2}\|x_1 - x_0\|^2 \\ &= \left(\|y^\delta - F(x_0)\| - \frac{\sqrt{C}}{\sqrt{2}}\|x_1 - x_0\|\right)^2 \geq 0 . \end{aligned} \quad (3.8)$$

In particular, it follows for $\alpha < 1$

$$\begin{aligned}\alpha\|x_1 - \bar{x}\|^2 &\overset{(3.8)}{\leq} \tilde{J}_\alpha(x_1, x_0) = \min_x \tilde{J}_\alpha(x, x_0) \leq \tilde{J}_\alpha(x_0, x_0) \\ &= \|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2 ,\end{aligned}$$

i.e.

$$\|x_1 - \bar{x}\|^2 \leq \frac{\|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2}{\alpha} = r^2 ,$$

and for $\alpha \geq 1$

$$\begin{aligned}\|x_1 - \bar{x}\|^2 \leq \alpha\|x_1 - \bar{x}\|^2 &\overset{(3.8)}{\leq} \tilde{J}_\alpha(x_1, x_0) \leq \tilde{J}_\alpha(x_0, x_0) \\ &= \|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2 = r^2 ,\end{aligned}$$

and thus $x_1 \in K_r(\bar{x})$.

By Lemma 1 it follows that $C\|x_1 - x_0\|^2 - \|F(x_1) - F(x_0)\|^2 \geq 0$ and $J_\alpha(x) \leq \tilde{J}_\alpha(x, x_0)$, and we get

$$\|y^\delta - F(x_1)\|^2 \leq J_\alpha(x_1) \leq \tilde{J}_\alpha(x_1, x_0) \leq \tilde{J}_\alpha(x_0, x_0) \leq \|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2 \ ,$$

and combining this estimate with the definition of C in (3.2) yields

$$2L\|y^\delta - F(x_1)\| \leq 2L\sqrt{\|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2} \leq C \ . \tag{3.9}$$

Now let us assume that the following properties hold for all $i = 1, \cdots k-1$:

$$x_i \in K_r(\bar{x}) \tag{3.10}$$
$$C\|x_i - x_{i-1}\| - \|F(x_i) - F(x_{i-1})\| \geq 0 \tag{3.11}$$
$$2L\|y^\delta - F(x_i)\| \leq C \ , \tag{3.12}$$

where x_i denotes a minimizer of the functional $\tilde{J}(x, x_{i-1})$. For $i = 1$, these properties have already been shown. As for the case $i = 1$, we have to show that the functional $\tilde{J}_\alpha(x, x_{k-1})$ has a minimizer. First, we show that it is bounded from below: As in (3.5) we get

$$\tilde{J}_\alpha(x, x_{k-1}) - \alpha\|x - \bar{x}\|^2 \geq \|y^\delta - F(x_{k-1})\|^2 - 2\|y^\delta - F(x_{k-1})\|\|F(x_{k-1}) - F(x)\| + C\|x - x_{k-1}\|^2 \tag{3.13}$$

By Taylor expansion, we get

$$\|F(x_{k-1}) - F(x)\| \leq \|F'(x_{k-1})\|\|x_{k-1} - x\| + \frac{L}{2}\|x_{k-1} - x\|^2 \ . \tag{3.14}$$

Now let us assume that $\tilde{J}_\alpha(x, x_{k-1})$ is not bounded from below. As F is continuous, there exists a sequence $\{x_l\}_{l\in\mathbb{N}}$ with $\|x_l\| \to \infty$ and $\tilde{J}_\alpha(x_l, x_{k-1}) \to -\infty$. In particular, for l large enough, follows from (3.14)

$$\|F(x_{k-1}) - F(x_l)\| \leq L\|x_{k-1} - x_l\|^2 \ ,$$

and combining this estimate with (3.13) yields

$$\tilde{J}_\alpha(x_l, x_{k-1}) - \alpha\|x_l - \bar{x}\|^2 \geq \|y^\delta - F(x_{k-1})\|^2 - 2L\|y^\delta - F(x_{k-1})\|\|x_l - x_{k-1}\|^2 + C\|x_l - x_{k-1}\|^2 \ .$$

From (3.12) follows $2L\|y^\delta - F(x_{k-1})\| \leq C$ and thus

$$\tilde{J}_\alpha(x_l, x_{k-1}) - \alpha\|x_l - \bar{x}\|^2 \geq \|y^\delta - F(x_{k-1})\|^2 \geq 0,$$

in contradiction to our assumption $\tilde{J}_\alpha(x_l, x_{k-1}) \to -\infty$, and thus $\tilde{J}_\alpha(x, x_{k-1})$ is bounded from below. By the same argument, we find $\tilde{J}_\alpha(x_l, x_{k-1}) \geq \alpha\|x_l - \bar{x}\|^2 \to \infty$, for any sequence x_l with $\|x_l\| \to \infty$ and thus the functional is coercive and has a minimizer x_k.
As in (3.13), we get by using (3.14)

$$\begin{aligned}
\tilde{J}_\alpha(x_k, x_{k-1}) - \alpha\|x_k - \bar{x}\|^2 \ &\geq \ \|y^\delta - F(x_{k-1})\|^2 + 2\langle y^\delta - F(x_{k-1}), F(x_{k-1}) - F(x_k)\rangle \\
&\qquad\qquad + C\|x_k - x_{k-1}\|^2 \\
&\geq \ \|y^\delta - F(x_{k-1})\|^2 - 2\|y^\delta - F(x_{k-1})\|\|F'(x_{k-1})\|\|x_k - x_{k-1}\| \\
&\quad - L\|y^\delta - F(x_{k-1})\|\|x_k - x_{k-1}\|^2 + C\|x_k - x_{k-1}\|^2 \ .
\end{aligned}$$

By (3.2) and assumption (3.12) we have $C/2 \geq L\|y^\delta - F(x_{k-1})\|$, and thus

$$\begin{aligned}\tilde{J}_\alpha(x_k, x_{k-1}) - \alpha\|x_k - \bar{x}\|^2 &\geq \|y^\delta - F(x_{k-1})\|^2 - 2\|y^\delta - F(x_{k-1})\|\|F'(x_{k-1})\|\|x_k - x_{k-1}\| \\ &\quad + \frac{C}{2}\|x_k - x_{k-1}\|^2 .\end{aligned}$$

As $x_{k-1} \in K_r(\bar{x})$, it follows from (3.2) that $\|F'(x_{k-1})\| \geq \sqrt{C/2}$ holds, and we get finally

$$\begin{aligned}\tilde{J}_\alpha(x_k, x_{k-1}) - \alpha\|x_l - \bar{x}\|^2 &\geq \|y^\delta - F(x_{k-1})\|^2 - 2\frac{\sqrt{C}}{\sqrt{2}}\|y^\delta - F(x_{k-1})\|\|x_k - x_{k-1}\| \\ &\quad + \frac{C}{2}\|x_k - x_{k-1}\|^2 \\ &= \left(\|y^\delta - F(x_{k-1})\| - \frac{\sqrt{C}}{\sqrt{2}}\|x_k - x_{k-1}\|\right)^2 \geq 0 . \qquad (3.15)\end{aligned}$$

In particular, it follows for $\alpha < 1$ by assumption (3.11)

$$\begin{aligned}\alpha\|x_k - \bar{x}\|^2 &\overset{(3.15)}{\leq} \tilde{J}_\alpha(x_k, x_{k-1}) = \min_x \tilde{J}_\alpha(x, x_{k-1}) \leq \tilde{J}_\alpha(x_{k-1}, x_{k-1}) \\ &= \|y^\delta - F(x_{k-1})\|^2 + \alpha\|x_{k-1} - \bar{x}\|^2 \\ &\leq \|y^\delta - F(x_{k-1})\|^2 + \alpha\|x_{k-1} - \bar{x}\|^2 + C\|x_{k-1} - x_{k-2}\|^2 - \|F(x_{k-1}) - F(x_{k-2})\|^2 \\ &= \tilde{J}_\alpha(x_{k-1}, x_{k-2}) \leq \tilde{J}_\alpha(x_{k-2}, x_{k-2}) \leq \cdots \leq \tilde{J}_\alpha(x_0, x_0) \\ &= \|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2\end{aligned}$$

i.e.

$$\|x_k - \bar{x}\|^2 \leq \frac{\|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2}{\alpha} \leq r^2 ,$$

and in the same way follows for $\alpha \geq 1$

$$\begin{aligned}\|x_k - \bar{x}\|^2 \leq \alpha\|x_k - \bar{x}\|^2 &\overset{(3.15)}{\leq} \tilde{J}_\alpha(x_k, x_{k-1}) \leq \tilde{J}_\alpha(x_{k-1}, x_{k-1}) \leq \cdots \leq \tilde{J}_\alpha(x_0, x_0) \\ &= \|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2 \leq r^2 ,\end{aligned}$$

and thus $x_k \in K_r(\bar{x})$.
As in Lemma 1, it follows $C\|x_k - x_{k-1}\|^2 - \|F(x_k) - F(x_{k-1})\|^2 \geq 0$ and $J_\alpha(x) \leq \tilde{J}_\alpha(x, x_{k-1})$, and we get

$$\begin{aligned}\|y^\delta - F(x_k)\|^2 &\leq J_\alpha(x_k) \leq \tilde{J}_\alpha(x_k, x_{k-1}) \leq \tilde{J}_\alpha(x_{k-1}, x_{k-1}) \leq \cdots \leq \tilde{J}_\alpha(x_0, x_0) \\ &= \|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2 , \qquad (3.16)\end{aligned}$$

and combining this estimate with the definition of C (3.2) yields

$$2L\|y^\delta - F(x_k)\| \leq 2L\sqrt{\|y^\delta - F(x_0)\|^2 + \alpha\|x_0 - \bar{x}\|^2} \leq C , \qquad (3.17)$$

i.e. we have shown that the assumptions (3.10)-(3.12) hold also for $i = k$. $\square$

Corollary 3 *The sequences of functionals* $\{J_\alpha(x_k)\}_{k=0,1,2,\dots}$ *and* $\{\tilde{J}_\alpha(x_{k+1},x_k)\}_{k=0,1,2,\dots}$ *are non-increasing.*

Proof. This follows now by $J_\alpha(x_{k+1}) \le \tilde{J}_\alpha(x_{k+1},x_k) \le \tilde{J}_\alpha(x_k,x_k) = J_\alpha(x_k) \le \tilde{J}_\alpha(x_k,x_{k-1})$. □

4 On the minimization of the replacement functional

In this section, we elaborate necessary conditions for a minimizer of the functional $\tilde{J}_\alpha(x,x_{k-1})$. Moreover, we prove that $\tilde{J}_\alpha(x,x_k)$ is globally convex for each $k = 0,1,2,\dots$.

Lemma 4 *The derivative* $\tilde{J}'_\alpha(x,a)h$ *of* $\tilde{J}_\alpha(x,a)$ *is given by*

$$\tilde{J}'_\alpha(x,a)h = -2\langle F'(x)^*(y^\delta - F(a)) + (Ca + \alpha\bar{x}) - (C+\alpha)x, h\rangle \ . \tag{4.1}$$

Proof. It is

$$\tilde{J}_\alpha(x+h,a) = \|y^\delta - F(x+h)\|^2 + \alpha\|x-\bar{x}+h\|^2 + C\|x-a+h\|^2 - \|F(x+h)-F(a)\|^2 \ .$$

By Taylor's expansion, $F(x+h) = F(x) + F'(x)h + O(\|h\|^2)$, we get

$$\begin{aligned} \tilde{J}_\alpha(x+h,a) &= \|y^\delta - F(x) - F'(x)h + O(\|h\|^2)\|^2 + \alpha\|x-\bar{x}+h\|^2 + C\|x-a+h\|^2 \\ &\quad -\|F(x) - F(a) + F'(x)h + O(\|h\|^2)\|^2 \\ &= \|y^\delta - F(x)\|^2 + \|F'(x)h\|^2 - 2\langle y^\delta - F(x), F'(x)h\rangle \\ &\quad +\alpha(\|x-\bar{x}\|^2 + \|h\|^2 + 2\langle x-\bar{x},h\rangle) + C(\|x-a\|^2 + \|h\|^2 + 2\langle x-a,h\rangle) \\ &\quad -(\|F(x)-F(a)\|^2 + \|F'(x)h\|^2 + 2\langle F(x)-F(a), F'(x)h\rangle) + O(\|h\|^2) \ . \end{aligned}$$

It follows

$$\begin{aligned} \frac{\tilde{J}_\alpha(x+h,a) - \tilde{J}_\alpha(x,a)}{2} &= -\langle F'(x)^*(y^\delta - F(x)), h\rangle + \alpha\langle x-\bar{x},h\rangle + C\langle x-a,h\rangle \\ &\quad -\langle F'(x)^*(F(x)-F(a)),h\rangle + O(\|h\|^2) \\ &= -\langle F'(x)^*(y^\delta - F(a)) - \alpha(x-\bar{x}) - C(x-a), h\rangle + O(\|h\|^2) \\ &= -\langle F'(x)^*(y^\delta - F(a)) - (C+\alpha)x + \alpha\bar{x} + Ca, h\rangle + O(\|h\|^2) \ . \end{aligned}$$

and thus the derivative is given by (4.1). □

The necessary condition for a minimum of (2.2) thus reads as

$$x = \underbrace{\frac{1}{C+\alpha}\left(F'(x)^*(y^\delta - F(a)) + \alpha\bar{x} + Ca\right)}_{=:\Phi_\alpha(x,a)} \ . \tag{4.2}$$

To minimize (2.2), we will use a fixed point iteration for $\Phi_\alpha(x,a)$. As $\tilde{J}_\alpha(x,a)$ has by Proposition 2 a minimizer, (4.2) has at least one fixed point. It remains to show that $\Phi_\alpha(x,a)$ is a contraction operator:

Lemma 5 *The operator $\Phi_\alpha(x,a)$ is a contraction, i.e. $\|\Phi_\alpha(x,a) - \Phi_\alpha(\tilde{x},a)\| \le q\|x-\tilde{x}\|$, if*

$$q := \frac{L}{C+\alpha}\sqrt{J_\alpha(a)} < 1 \ .$$

Proof. We have $\Phi_\alpha(x,a) - \Phi_\alpha(\tilde{x},a) = \frac{1}{C+\alpha}(F'(x) - F'(\tilde{x}))^*(y^\delta - F(a))$, and by using the Lipschitz–continuity of F' we get

$$\begin{aligned} \|\Phi_\alpha(x,a) - \Phi_\alpha(\tilde{x},a)\| &= \frac{1}{C+\alpha}\|F'(x) - F'(\tilde{x})\|\|y^\delta - F(a)\| \\ &\le \frac{L}{C+\alpha}\|y^\delta - F(a)\|\|x-\tilde{x}\| \le \frac{L}{C+\alpha}\sqrt{J_\alpha(a)}\|x-\tilde{x}\| \ . \end{aligned}$$

□

Proposition 6 *In our algorithm, the operator $\Phi_\alpha(x,x_k)$ is for all $k = 0,1,2,\ldots$ and all $\alpha \ge 0$ a contraction.*

Proof. By the definition of C in (3.2), Lemma 5 (setting $a = x_0$), we deduce that $\Phi_\alpha(x,x_0)$ is a contraction with

$$q = \frac{L}{C+\alpha}\sqrt{J_\alpha(x_0)} = \frac{C}{2(C+\alpha)} \le \frac{1}{2} < 1.$$

With the help of Corollary 3, we complete the proof

$$\|\Phi_\alpha(x,x_k) - \Phi_\alpha(\tilde{x},x_k)\| \le \frac{L}{C+\alpha}\sqrt{J_\alpha(x_k)} \le \frac{L}{C+\alpha}\sqrt{J_\alpha(x_{k-1})} \le \ldots \frac{L}{C+\alpha}\sqrt{J_\alpha(x_0)} \ .$$

□

Up to here, we do know that our fixed point iteration for (4.2) converges towards a critical point of $\tilde{J}_\alpha(x,x_k)$.

Proposition 7 *The necessary equation (4.2) for a minimum of the functional $\tilde{J}_\alpha(x,x_k)$ has a unique fixed point, and the fixed point iteration converges towards the minimizer.*

Proof. To prove this Proposition, we have to investigate the Taylor expansion of $\tilde{J}_\alpha$ more closely. By Taylor's expansion for F and the Lipschitz–continuity of F' we get

$$F(x+h) = F(x) + F'(x)h + R(x,h) \tag{4.3}$$

with

$$\|R(x,h)\| \le \frac{L}{2}\|h\|^2 \ . \tag{4.4}$$

As in the proof of Lemma 4 we get

$$\begin{aligned} \tilde{J}_\alpha(x+h,x_k) &= \tilde{J}_\alpha(x,x_k) + \tilde{J}'_\alpha(x,x_k)h - 2\langle y^\delta - F(x), R(x,h)\rangle - 2\langle F(x) - F(x_k), R(x,h)\rangle \\ &\quad +(\alpha+C)\|h\|^2 \\ &= \tilde{J}_\alpha(x,x_k) + \tilde{J}'_\alpha(x,x_k)h - 2\langle y^\delta - F(x_k), R(x,h)\rangle + (\alpha+C)\|h\|^2 \ , \end{aligned} \tag{4.5}$$

and by using $C \geq 2L\|y^\delta - F(x_k)\|$ follows

$$\begin{aligned} -2\langle y^\delta - F(x_k), R(x,h)\rangle + (\alpha + C)\|h\|^2 &\geq -2\|y^\delta - F(x_k)\|\|R(x,h)\| + (\alpha+C)\|h\|^2 \\ &\geq (-L\|y^\delta - F(x_k)\| + \alpha + C)\|h\|^2 \\ &\geq (C/2+\alpha)\|h\|^2 \ . \end{aligned} \tag{4.6}$$

Now assume $\tilde{x}$ is a critical point of $\tilde{J}_\alpha$, i.e. $\tilde{J}'_\alpha(\tilde{x}, x_k)h = 0$ for all h. Consequently, by (4.5), (4.6) we have

$$\tilde{J}_\alpha(\tilde{x}+h, x_k) \geq \tilde{J}_\alpha(\tilde{x}, x_k) + (C/2+\alpha)\|h\|^2 \ ,$$

and in particular

$$\tilde{J}_\alpha(\tilde{x}+h, x_k) > \tilde{J}_\alpha(\tilde{x}, x_k) \qquad \text{for all } h \neq 0 \ . \tag{4.7}$$

Thus, every critical point is a global minimizer of $\tilde{J}_\alpha(x, x_k)$, and, again by (4.7), there exists only one global minimizer. □

By assuming more regularity on F it is possible to sharpen the above given statement:

Proposition 8 *Let F be a twice continuously differentiable operator. Then the functional $\tilde{J}_\alpha(x, x_k)$ is strictly convex.*

Proof. With a slight abuse of notation we set $\tilde{J}_\alpha(x) := \tilde{J}_\alpha(x, x_k)$. By (4.5) we have

$$\tilde{J}_\alpha(x+h) = \tilde{J}_\alpha(x) + \tilde{J}'_\alpha(x)h + g_\alpha(x,h) \ , \tag{4.8}$$

where $g_\alpha(x,h)$ is defined by

$$g_\alpha(x,h) := -2\langle y^\delta - F(x_k), R(x,h)\rangle + (\alpha+C)\|h\|^2 \ . \tag{4.9}$$

For strict convexity, we have to show that

$$\tilde{J}_\alpha((1-\lambda)x_1 + \lambda x_2) < (1-\lambda)\tilde{J}_\alpha(x_1) + \lambda \tilde{J}_\alpha(x_2)$$

holds for $\lambda \in (0,1)$ and arbitrary x_1, x_2. We have

$$\begin{aligned} \tilde{J}_\alpha((1-\lambda)x_1 + \lambda x_2)) &= \tilde{J}_\alpha(x_1 + \lambda(x_2 - x_1)) = \tilde{J}_\alpha(x_2 + (1-\lambda)(x_1 - x_2)) \\ &= (1-\lambda)\tilde{J}_\alpha(x_1 + \lambda(x_2-x_1)) + \lambda \tilde{J}_\alpha(x_2 + (1-\lambda)(x_1-x_2)) \end{aligned} \tag{4.10}$$

and with

$$\begin{aligned} \tilde{J}_\alpha(x_1 + \lambda(x_2 - x_1)) &= \tilde{J}_\alpha(x_1) + \lambda \tilde{J}'_\alpha(x_1)(x_2 - x_1) + g_\alpha(x_1, \lambda(x_2 - x_1)) \\ \tilde{J}_\alpha(x_2 + (1-\lambda)(x_1 - x_2)) &= \tilde{J}_\alpha(x_2) + (1-\lambda)\tilde{J}'_\alpha(x_2)(x_1 - x_2) + g_\alpha(x_2, (1-\lambda)(x_1 - x_2)) \end{aligned}$$

we obtain

$$\begin{aligned} \tilde{J}_\alpha((1-\lambda)x_1 + \lambda x_2)) &= (1-\lambda)\tilde{J}_\alpha(x_1) + \lambda\tilde{J}_\alpha(x_2) + \lambda(1-\lambda)\left[\tilde{J}'_\alpha(x_1) - \tilde{J}'_\alpha(x_2)\right](x_2 - x_1) \\ &\quad + (1-\lambda)g_\alpha(x_1, \lambda(x_2-x_1)) + \lambda g_\alpha(x_2, (1-\lambda)(x_1-x_2)) \ . \end{aligned}$$

Thus $\tilde{J}_\alpha$ is strict convex if for all $\lambda \in (0,1)$

$$\begin{aligned} D(x_1,x_2,\lambda) &:= \lambda(1-\lambda)\left[\tilde{J}'_\alpha(x_1) - \tilde{J}'_\alpha(x_2)\right](x_2-x_1) \\ &\quad +(1-\lambda)g_\alpha(x_1,\lambda(x_2-x_1)) + \lambda g_\alpha(x_2,(1-\lambda)(x_1-x_2)) < 0 \ . \end{aligned}$$

We have

$$\begin{aligned} \frac{\tilde{J}'_\alpha(x_1) - \tilde{J}'_\alpha(x_2)}{2}(x_2-x_1) &= -\langle F'(x_1)^*(y^\delta - F(x_k)) + Cx_k + \alpha\bar{x} - (C+\alpha)x_1, x_2-x_1\rangle \\ &\quad +\langle F'(x_2)^*(y^\delta - F(x_k)) + Cx_k + \alpha\bar{x} - (C+\alpha)x_2, x_2-x_1\rangle \\ &= -(C+\alpha)\|x_2-x_1\|^2 \\ &\qquad -\langle (F'(x_1)-F'(x_2))^*(y^\delta - F(x_k)), x_2-x_1\rangle \ . \\ &= -(C+\alpha)\|x_2-x_1\|^2 - \langle y^\delta - F(x_k), F'(x_1) - F'(x_2)(x_2-x_1)\rangle. \end{aligned}$$

As F is twice continuously Fréchet differentiable, it is

$$F'(x_1) = F'(x_2) + \int_0^1 F''(x_2 + \tau(x_1-x_2))(x_1-x_2,\cdot)\,d\tau$$

and thus,

$$\begin{aligned} \left[\tilde{J}'_\alpha(x_1) - \tilde{J}'_\alpha(x_2)\right](x_2-x_1) &= -2(C+\alpha)\|x_2-x_1\|^2 + \\ &\quad 2\langle y^\delta - F(x_k), \int_0^1 F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2 d\tau\rangle, \end{aligned} \tag{4.11}$$

where we have used the shorthand $F''(\cdot)(h,h) = F''(\cdot)(h)^2$. Again, as F is twice continuously Fréchet-differentiable, the function $R(x,h)$ in (4.9) is given by

$$R(x,h) = \int_0^1 (1-\tau)F''(x+\tau h)h^2\,d\tau \ ,$$

and thus we obtain

$$\begin{aligned} R(x_1,\lambda(x_2-x_1)) &= \lambda^2 \int_0^1 (1-\tau)F''(x_1+\tau\lambda(x_2-x_1))(x_2-x_1)^2\,d\tau \\ &= \int_{1-\lambda}^1 (\tau-(1-\lambda))F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2\,d\tau \end{aligned} \tag{4.12}$$

and in the same way

$$R(x_2,(1-\lambda)(x_1-x_2)) = \int_0^{1-\lambda} (1-\lambda-\tau)F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2\,d\tau \ . \tag{4.13}$$

Combining definition (4.9) and equations (4.11), (4.12) and (4.13) yields

$$D(x_1,x_2,\lambda) = -\lambda(1-\lambda)(C+\alpha)\|x_1-x_2\|^2 + 2\langle y^\delta - F(x_k), f(x_1,x_2,\lambda)\rangle \ , \tag{4.14}$$

where

$$\begin{aligned} f(x_1,x_2,\lambda) \quad := \quad & \lambda(1-\lambda)\int_0^1 F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2\,d\tau \\ & -(1-\lambda)\int_{1-\lambda}^1 (\tau-(1-\lambda))F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2\,d\tau \\ & -\lambda\int_0^{1-\lambda}(1-\lambda-\tau)F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2\,d\tau \ . \end{aligned}$$

The functional $f(x_1,x_2,\lambda)$ can now be recasted as follows

$$\begin{aligned} f(x_1,x_2,\lambda) \quad = \quad & \lambda(1-\lambda)\int_0^{1-\lambda} F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2\,d\tau \\ & +\lambda(1-\lambda)\int_{1-\lambda}^1 F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2\,d\tau \\ & -(1-\lambda)\int_{1-\lambda}^1 (\tau-(1-\lambda))F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2\,d\tau \\ & -\lambda\int_0^{1-\lambda}(1-\lambda-\tau)F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2\,d\tau \\ = \quad & \lambda\int_0^{1-\lambda}\tau F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2\,d\tau \\ & +(1-\lambda)\int_{1-\lambda}^1 (1-\tau)F''(x_2+\tau(x_1-x_2))(x_1-x_2)^2\,d\tau. \end{aligned}$$

In order to estimate $\|f(x_1,x_2,\lambda)\|$ it is necessary to estimate the integrals separately. Due to the Lipschitz–continuity of the first derivative, the second derivative can be globally estimated by $\|F''(x)\| \le L$, and it follows

$$\begin{aligned}
\lambda \left\| \int_0^{1-\lambda} \tau F''(x_2 + \tau(x_1 - x_2))(x_1 - x_2)^2 \, d\tau \right\| &\le \lambda \frac{(1-\lambda)^2}{2} L \|x_1 - x_2\|^2 \ , \\
(1-\lambda) \left\| \int_{1-\lambda}^{1} (1-\tau) F''(x_2 + \tau(x_1 - x_2))(x_1 - x_2)^2 \, d\tau \right\| &\le (1-\lambda) \frac{\lambda^2}{2} L \|x_1 - x_2\|^2
\end{aligned}$$

and thus

$$\|f(x_1, x_2, \lambda)\| \le \frac{\lambda(1-\lambda)}{2} L \|x_1 - x_2\|^2 \ . \tag{4.15}$$

Combining (4.14) and (4.15) yields for $\lambda \in (0,1)$

$$\begin{aligned}
D(x_1, x_2, \lambda) &\le -\lambda(1-\lambda)(C+\alpha)\|x_1 - x_2\|^2 + 2\|y^\delta - F(x_k)\| \|f(x_1, x_2, \lambda)\| \\
&\le -\lambda(1-\lambda)(C+\alpha)\|x_1 - x_2\|^2 + \frac{\lambda(1-\lambda)}{2} 2L\|y^\delta - F(x_k)\| \|x_1 - x_2\|^2 \\
&\overset{(3.17)}{\le} -\lambda(1-\lambda)\left(\frac{C}{2} + \alpha\right)\|x_1 - x_2\|^2 \le -\lambda(1-\lambda)\frac{C}{2}\|x_1 - x_2\|^2 < 0 \ ,
\end{aligned}$$

and thus the functional is strictly convex. □

5 Convergence properties of the proposed iteration

Within this section we aim to show that the sequence of iterates x_k converges strongly towards a minimizer of the Tikhonov functional. To achieve norm convergence, we prove some preliminary Lemmas.

Lemma 9 *The sequence of iterates $\{x_k\}_{k=0,1,2,\ldots}$ has a weakly convergent subsequence.*

Proof. This is an immediate consequence of Proposition 2, in which it is shown that for $k = 0, 1, 2, \ldots$ the iterates x_k are contained in $K_r(\bar{x})$, i.e. $\|x_{k+1} - \bar{x}\|_X \le r$ or equivalently $\|x_{k+1}\|_X \le r + \|\bar{x}\|_X < \infty$. Since the iterates are uniformly bounded, we deduce that there exists at least one accumulation point $x_\alpha^\star$ with $x_{k,l} \xrightarrow{w} x_\alpha^\star$, where $x_{k,l}$ denotes a subsequence of x_k. □

Lemma 10 *The sequence $\{\|x_{k+1} - x_k\|\}_{k=0,1,2,\ldots}$ converges to zero.*

Proof. With the help of Corollary 3, we observe that

$$\begin{aligned}
0 &\le \sum_k^N \left\{ C\|x_{k+1} - x_k\|^2 - \|F(x_{k+1}) - F(x_k)\|^2 \right\} \\
&\le \sum_k^N \left\{ \tilde{J}_\alpha(x_{k+1}, x_k) - J_\alpha(x_{k+1}) \right\} \le \sum_k^N \left\{ J_\alpha(x_k) - J_\alpha(x_{k+1}) \right\} \\
&= J_\alpha(x_0) - J_\alpha(x_{N+1}) \le J_\alpha(x_0) \ ,
\end{aligned}$$

i.e. the finite sums are uniformly bounded (independent on N). By the Taylor expansion of F, we have

$$\|F(x_{k+1}) - F(x_k)\| \le \int_0^1 \|F'(x_k + \tau(x_{k+1} - x_k))\| \|x_{k+1} - x_k\| \, d\tau \le \frac{C}{2} \|x_{k+1} - x_k\| \ ,$$

and thus

$$0 \le \frac{C}{2} \|x_{k+1} - x_k\|^2 \le C \|x_{k+1} - x_k\|^2 - \|F(x_{k+1}) - F(x_k)\|^2 \longrightarrow 0$$

as $k \to \infty$ and the assertion follows. $\square$

Lemma 11 *Every subsequence of x_k has a convergent subsequence $x_{k,l}$ that converges strongly towards a function $x_\alpha^\star$, and $x_\alpha^\star$ satisfies the necessary condition for a minimizer of the Tikhonov functional:*

$$\alpha(x_\alpha^\star - \bar{x}) = F'(x_\alpha^\star)^* (y^\delta - F(x_\alpha^\star)) \ . \tag{5.1}$$

Proof. According to (4.2), the minimizer x_{k+1} of $\tilde{J}_\alpha(x, x_k)$ fulfills

$$x_{k+1} = \frac{1}{C + \alpha} \left(C x_k + F'(x_{k+1})^* (y^\delta - F(x_k)) + \alpha \bar{x} \right) \ .$$

Thus,

$$x_{k+1} - x_k = -\frac{\alpha}{\alpha + C} x_k + \frac{1}{C + \alpha} \left(F'(x_k)^* (y^\delta - F(x_k)) + \alpha \bar{x} + \left(F'(x_{k+1}) - F'(x_k) \right)^* (y^\delta - F(x_k)) \right) \tag{5.2}$$

and, moreover, by Lemma 10, $\|x_{k+1} - x_k\| \to 0$, and thus

$$\| \left(F'(x_{k+1}) - F'(x_k) \right)^* (y^\delta - F(x_k)) \| \le L \|x_k - x_{k+1}\| \|y^\delta - F(x_0)\| \to 0 \ .$$

It follows by taking the limit $k \to \infty$ in (5.2),

$$0 = \lim_{k \to \infty} \left(\alpha(\bar{x} - x_k) + F'(x_k)^* (y^\delta - F(x_k)) \right) \ . \tag{5.3}$$

As the sequence x_k is bounded, every subsequence has a weakly convergent subsequence. Let $x_{k,l}$ be an arbitrary weakly convergent subsequence with weak limit $x_\alpha^\star$ (for simplicity, we will denote this sequence by x_k, too). Since

$$F'(x_k)^* (y^\delta - F(x_k)) = F'(x_k)^* (y^\delta - F(x_\alpha^\star)) + F'(x_k)^* (F(x_\alpha^\star) - F(x_k)) \ ,$$

and because of $\|F'(x_k)^* (F(x_\alpha^\star) - F(x_k))\| \le 2C \|F(x_\alpha^\star) - F(x_k)\| \to 0$ and by assumption (2.3), i.e. $F'(x_k)^* (y^\delta - F(x_\alpha^\star)) \to F'(x_\alpha^\star)^* (y^\delta - F(x_\alpha^\star))$, we consequently obtain

$$\lim_{k \to \infty} F'(x_k)^* (y^\delta - F(x_k)) = F'(x_\alpha^\star)^* (y^\delta - F(x_\alpha^\star)) \ . \tag{5.4}$$

Combining (5.4) with (5.2) proves that $x_{k,l}$ converges, and as $x_\alpha^\star$ is the weak limit of the sequence, $x_{k,l} \to x_\alpha^\star$. Equation (5.1) follows by taking the limit in (5.3). $\square$

In principle, the limits of different convergent subsequences of x_k can be different. Let $x_{k,l} \times^*_\alpha$ be a subsequence of x_k, and denote by $\tilde{x}_{k,l}$ the predecessor of $x_{k,l}$ in x_k, i.e. $x_{k,l} = x_i$ and $\tilde{x}_{k,l} = x_{i-1}$. Then we observe

$$\tilde{J}_\alpha(x_{k,l}, \tilde{x}_{k,l}) \to J_\alpha(x^*_\alpha)$$

Morover, as we have $\tilde{J}_\alpha(x_{k+1,x_k}) \leq J_\alpha(x_k, x_{k-1})$ for all k, it turn out that the value of the Tikhonov functional for every limit x^*_α of a convergent subsequence stays the same:

$$J_\alpha(x^*_\alpha) = const \ . \tag{5.5}$$

We will now give a simple criterion that ensures convergence of the whole sequence x_k.

Theorem 12 *Assume that there exists at least one isolated limit x^*_α of a subsequence $x_{k,l}$ of x_k. Then $x_k \longrightarrow x^*_\alpha$ holds.*

Proof. By x^*_α we will denote the isolated limit of the sequence $x_{k,l}$. Let M denote the set of all limits of subsequences of the sequence $\{x_k\}$, and $M_1 := M \setminus \{x^*_\alpha\}$. Setting $r = dist(x^*_\alpha, M_1)/2$, we define

$$\begin{aligned} B_r &:= \{x_k : \|x_k - x^*_\alpha\| \leq r\} \\ \bar{B}_r &:= \{x_k : x_k \notin K_r\} \ . \end{aligned}$$

Now let us assume $M_1 \neq \emptyset$. Then both B_r, $\bar{B}_r$ contain infinitely many elements. In particular, there exist infinitely many pairs of iterates x_k, x_{k+1} with $x_k \in K_r$ and $x_{k+1} \in \bar{B}_r$, and we can define a subseqence $\tilde{x}_k$ by picking all pairs $x_k \in B_r$ and $x_{k+1} \in \bar{B}_r$ out of the sequence $\{x_k\}_{k\in\mathbb{N}}$, i.e.

$$\begin{aligned} \tilde{x}_{2l} &= x_k \in B_r \\ \tilde{x}_{2l+1} &= x_{k+1} \in \bar{B}_r \end{aligned} \tag{5.6}$$

Because of Lemma 10 we observe $\|x_{2l} - x_{2l+1}\| \to 0$, and with (5.6) follows that the elements of $\tilde{x}_l$ come arbitrary close to $\partial B_r = \{x : \|x - x^*_\alpha\| = r\}$, i.e.

$$\lim_{l\to\infty} \|\tilde{x}_l - x^*_\alpha\| = r \ . \tag{5.7}$$

According to Lemma 11, every subsequence of x_k has a convergent subseqence. Let $\tilde{x}_{l,k}$ be a convergent subsequence of $\tilde{x}_l$ with limit $\tilde{x}^*_\alpha$. Because of (5.7) holds $\tilde{x}^*_\alpha \in \partial B_r$. On the other hand, as $x^*_\alpha \neq \tilde{x}^*_\alpha$, we have $\tilde{x}^*_\alpha \in M_1$, which is a contradiction to $dist(x^*_\alpha, M_1) = 2r$.

We conclude $M_1 = \emptyset$, i.e. x^*_α is the only limit of convergent subsequences of x_k. As by Lemma 11 every subsequence of x_k has a subsequence that converges towards x^*_α, the whole sequence converges towards x^*_α by the convergence principles. □

On the other hand, we conclude the sequence x_k can only not converge if the Tikhonov functional has a dense set of critical points, and the belonging functional values are constant.

By the following Proposition, the fixed point $x^\star_\alpha$ is also a minimizer for the functional $\tilde{J}_\alpha(x, x^\star_\alpha)$.

Proposition 13 *The accumulation point $x_\alpha^\star$ is a minimizer for the functional $\tilde{J}_\alpha(x, x_\alpha^\star)$.*

Proof. We aim to show that for all $h \in X$,

$$\tilde{J}_\alpha(x_\alpha^\star + h, x_\alpha^\star) \geq \tilde{J}_\alpha(x_\alpha^\star, x_\alpha^\star) + \frac{C}{2}\|h\|^2 \ .$$

This is obtained by making use of

$$\tilde{J}_\alpha(x_\alpha^\star + h, x_\alpha^\star) = \tilde{J}_\alpha(x_\alpha^\star, x_\alpha^\star) + 2\langle y^\delta - F(x_\alpha^\star), F(x_\alpha^\star) - F(x_\alpha^\star + h)\rangle + 2\langle \alpha(x_\alpha^\star - \bar{x}), h\rangle + (\alpha + C)\|h\|^2$$

and

$$\alpha(x_\alpha^\star - \bar{x}) = F'(x_\alpha^\star)^*(y^\delta - F(x_\alpha^\star)) \ .$$

With the Lipschitz–continuity of F' this results in

$$\begin{aligned}
\tilde{J}_\alpha(x_\alpha^\star + h, x_\alpha^\star) &\geq \tilde{J}_\alpha(x_\alpha^\star, x_\alpha^\star) - 2\|y^\delta - F(x_\alpha^\star)\|\|F(x_\alpha^\star) - F(x_\alpha^\star + h) + F'(x_\alpha^\star)h\| + (\alpha + C)\|h\|^2 \\
&\geq \tilde{J}_\alpha(x_\alpha^\star, x_\alpha^\star) - 2\frac{C}{2L}\frac{L}{2}\|h\|^2 + (\alpha + C)\|h\|^2 \\
&= \tilde{J}_\alpha(x_\alpha^\star, x_\alpha^\star) + \frac{C}{2}\|h\|^2 + \alpha\|h\|^2 \geq \tilde{J}_\alpha(x_\alpha^\star, x_\alpha^\star) + \frac{C}{2}\|h\|^2 \ .
\end{aligned}$$

□

Equation (5.1) states that our algorithm reconstructs at least a critical point of the Tikhonov functional. In general, a critical point will not always be a minimizer of the Tikhonov functional. However, we will give a condition that ensures this property. Namely, if we impose the condition (2.6) and do assume that the solution $x^\dagger$ fulfills a smoothness condition, then we can show that every critical point of the Tikhonov functional is a global minimizer. We wish to remark that (2.6) is a rather strong condition. However, conditions of this type have been used earlier, e.g. for Landweber iteration [11, 16] and for Levenberg-Marquardt iteration [9].

Theorem 14 *Let F be a twice Fréchet differentiable operator with* (2.6)*. If a smoothness condition*

$$x^\dagger - \bar{x} = F'(x^\dagger)^*\omega \ , \qquad L\|\omega\| < 1/3 \tag{5.8}$$

holds, and the regularization parameter is chosen with

$$\alpha = \delta/\eta \text{ and } \eta \leq \|\omega\| \tag{5.9}$$

then (5.1) *has a unique solution. Thus the minimizer of the Tikhonov-functional is unique, too.*

Proof. Let x_α^δ denote a global minimizer of the Tikhonov functional, and $x_\alpha^\star$ be a critical point. With $F(x_\alpha^\star) = F(x_\alpha^\delta) + F'(x_\alpha^\delta)(x_\alpha^\star - x_\alpha^\delta) + R(x_\alpha^\delta, x_\alpha^\star - x_\alpha^\delta)$ we obtain

$$\begin{aligned}
\|y^\delta - F(x_\alpha^\star)\|^2 - \|y^\delta - F(x_\alpha^\delta)\|^2 &= \|F(x_\alpha^\delta) - F(x_\alpha^\star)\|^2 + 2\langle y^\delta - F(x_\alpha^\delta), F(x_\alpha^\delta) - F(x_\alpha^\star)\rangle \\
&= \|F(x_\alpha^\delta) - F(x_\alpha^\star)\|^2 + 2\langle F'(x_\alpha^\delta)^*(y^\delta - F(x_\alpha^\delta)), x_\alpha^\delta - x_\alpha^\star\rangle - 2\langle y^\delta - F(x_\alpha^\delta), R(x_\alpha^\delta, x_\alpha^\star - x_\alpha^\delta)\rangle \\
&\overset{(5.1)}{=} \|F(x_\alpha^\delta) - F(x_\alpha^\star)\|^2 + 2\alpha\langle x_\alpha^\delta - \bar{x}, x_\alpha^\delta - x_\alpha^\star\rangle - 2\langle y^\delta - F(x_\alpha^\delta), R(x_\alpha^\delta, x_\alpha^\star - x_\alpha^\delta)\rangle \ .
\end{aligned}$$

Because of

$$\alpha\|x_\alpha^\star - \bar{x}\|^2 - \alpha\|x_\alpha^\delta - \bar{x}\|^2 = \alpha\|x_\alpha^\star - x_\alpha^\delta\|^2 + 2\alpha\langle x_\alpha^\star - x_\alpha^\delta, x_\alpha^\delta - \bar{x}\rangle$$

it follows that

$$\begin{aligned} J_\alpha(x_\alpha^\star) - J_\alpha(x_\alpha^\delta) &= \|y^\delta - F(x_\alpha^\star)\|^2 - \|y^\delta - F(x_\alpha^\delta)\|^2 + \alpha\|x_\alpha^\star - \bar{x}\|^2 - \alpha\|x_\alpha^\delta - \bar{x}\|^2 \\ &= \|F(x_\alpha^\delta) - F(x_\alpha^\star)\|^2 + \alpha\|x_\alpha^\star - x_\alpha^\delta\|^2 - 2\langle y^\delta - F(x_\alpha^\delta), R(x_\alpha^\delta, x_\alpha^\star - x_\alpha^\delta)\rangle. \end{aligned} \tag{5.10}$$

By the same argument, we get

$$J_\alpha(x_\alpha^\delta) - J_\alpha(x_\alpha^\star) = \|F(x_\alpha^\delta) - F(x_\alpha^\star)\|^2 + \alpha\|x_\alpha^\star - x_\alpha^\delta\|^2 - 2\langle y^\delta - F(x_\alpha^\star), R(x_\alpha^\star, x_\alpha^\delta - x_\alpha^\star)\rangle\ . \tag{5.11}$$

Now, adding (5.10) and (5.11) yields

$$\begin{aligned} 0 &= 2\|F(x_\alpha^\delta) - F(x_\alpha^\star)\|^2 + 2\alpha\|x_\alpha^\star - x_\alpha^\delta\|^2 \\ &\quad -2\langle y^\delta - F(x_\alpha^\delta), R(x_\alpha^\delta, x_\alpha^\star - x_\alpha^\delta)\rangle - 2\langle y^\delta - F(x_\alpha^\star), R(x_\alpha^\star, x_\alpha^\delta - x_\alpha^\star)\rangle\ . \end{aligned} \tag{5.12}$$

For twice continuous differentiable operators, the quadratic remainder of the Taylor series is given by

$$\begin{aligned} R(x_\alpha^\delta, x_\alpha^\star - x_\alpha^\delta) &= \int_0^1 (1-\tau)F''(x_\alpha^\delta + \tau(x_\alpha^\star - x_\alpha^\delta))(x_\alpha^\star - x_\alpha^\delta, x_\alpha^\star - x_\alpha^\delta)\, d\tau \\ R(x_\alpha^\star, x_\alpha^\delta - x_\alpha^\star) &= \int_0^1 (1-\tau)F''(x_\alpha^\star + \tau(x_\alpha^\delta - x_\alpha^\star))(x_\alpha^\delta - x_\alpha^\star, x_\alpha^\delta - x_\alpha^\star)\, d\tau\ . \end{aligned}$$

Setting $\tau = 1 - \tau'$ and $h = x_\alpha^\delta - x_\alpha^\star$ and we obtain

$$R(x_\alpha^\delta, x_\alpha^\star - x_\alpha^\delta) = \int_0^1 \tau' F''(x_\alpha^\star + \tau' h)(h,h)\, d\tau'$$

and thus

$$\begin{aligned} &\langle y^\delta - F(x_\alpha^\delta), R(x_\alpha^\delta, x_\alpha^\star - x_\alpha^\delta)\rangle = \langle y^\delta - F(x_\alpha^\delta), \int_0^1 \tau F''(x_\alpha^\star + \tau h)(h,h)\, d\tau\rangle \\ &\quad = \langle y^\delta - F(x_\alpha^\delta), \int_0^1 (\tau - 1)F''(x_\alpha^\star + \tau h)(h,h)\, d\tau\rangle + \langle y^\delta - F(x_\alpha^\delta), \int_0^1 F''(x_\alpha^\star + \tau h)(h,h)\, d\tau\rangle \\ &\quad = -\langle y^\delta - F(x_\alpha^\delta), R(x_\alpha^\star, h)\rangle + \langle y^\delta - F(x_\alpha^\delta), \int_0^1 F''(x_\alpha^\star + \tau h)(h,h)\, d\tau\rangle \end{aligned} \tag{5.13}$$

Inserting (5.13) in (5.12) yields

$$\begin{aligned} 0 &= 2\|F(x_\alpha^\delta) - F(x_\alpha^\star)\|^2 + 2\alpha\|x_\alpha^\star - x_\alpha^\delta\|^2 - 2\langle F(x_\alpha^\star) - F(x_\alpha^\delta), R(x_\alpha^\star, x_\alpha^\delta - x_\alpha^\star)\rangle \\ &\quad -2\langle y^\delta - F(x_\alpha^\delta), \int_0^1 F''(x_\alpha^\star + \tau h)(h,h)\, d\tau\rangle \\ &\geq 2\|F(x_\alpha^\delta) - F(x_\alpha^\star)\|^2 + 2\alpha\|x_\alpha^\star - x_\alpha^\delta\|^2 - 2\|F(x_\alpha^\star) - F(x_\alpha^\delta)\|\|R(x_\alpha^\star, x_\alpha^\delta - x_\alpha^\star)\| \\ &\quad -2\|y^\delta - F(x_\alpha^\delta)\|\|\int_0^1 F''(x_\alpha^\star + \tau h)(h,h)\, d\tau\| \end{aligned}$$

By (2.6) we conclude $\|R(x_\alpha^\star, x_\alpha^\delta - x_\alpha^\star)\| \leq \|F(x_\alpha^\delta) - F(x_\alpha^\star)\|$, and from the smoothness condition (5.8), see [8] p.246, it follows

$$\|y^\delta - F(x_\alpha^\delta)\| \leq \delta + 2\alpha\|\omega\| \overset{(5.9)}{\leq} 3\alpha\|\omega\| \ .$$

Altogether we get

$$\begin{aligned} 0 &\geq 2\|F(x_\alpha^\delta) - F(x_\alpha^\star)\|^2 + 2\alpha\|x_\alpha^\star - x_\alpha^\delta\|^2 - 2\|F(x_\alpha^\star) - F(x_\alpha^\delta)\|\|R(x_\alpha^\star, x_\alpha^\delta - x_\alpha^\star)\| \\ &\quad -2\|y^\delta - F(x_\alpha^\delta)\|\|\int_0^1 F''(x_\alpha^\star + \tau h)(h,h)\, d\tau\| \\ &\geq (2 - 6L\|\omega\|)\alpha\|x_\alpha^\star - x_\alpha^\delta\|^2 \geq 0 \ , \end{aligned}$$

and thus we have shown $(2-6L\|\omega\|)\alpha\|x_\alpha^\star - x_\alpha^\delta\|^2 = 0$, and because of (5.8) holds $x_\alpha^\star = x_\alpha^\delta$. $\square$

Conditions (2.6), (5.8) ensure the convergence of our algorithm towards the unique minimizer of the Tikhonov functional. Using a proper parameter choice rule for the regularization parameter gives convergence/convergence rates for Tikhonov regularization. We might recall a few well known parameter rules.

(I) Let F be a weakly sequentially closed operator, and the regularization parameter α chosen such that $\alpha(\delta) \to 0$ and $\delta^2/\alpha \to 0$ as $\delta \to 0$. Then every sequence $x_{\alpha_k}^{\delta_k}$ with δ_k has a convergent subsequence that converges toward an $\bar{x}$- minimum norm solution $x^\dagger$. In particular, if a smoothness condition (5.8) holds, and the regularization parameter is chosen by $\alpha = \delta/\eta$, $\eta \leq \|\omega\|$, then we obtain a convergence rate of $\mathcal{O}(\sqrt{\delta})$ [8].

(II) Let F be a Fréchet differentiable operator with (2.4). Moreover, assume that $x^\dagger$ fulfills a smoothness condition $x^\dagger - \bar{x} = (F'(x^\dagger)^* F'(x^\dagger))^\nu \omega$ for $\nu \in [1/2, 1]$ with $L\|\omega\| < 1/3$. If the parameter is chosen by $\alpha \sim \delta^{2/(2\nu+1)}$, then we obtain a convergence rate of $\mathcal{O}(\delta^{2\nu/(2\nu+1)})$ [8].

(III) (Morozov's discrepancy principle) Let F be a twice continuous differentiable operator with (2.4), and assume $x^\dagger - \bar{x} = F'(x^\dagger)^*\omega$ with $L\|\omega\| \leq 0.241$. Then there exists a regularization parameter $\alpha \leq \delta/\eta$, $\eta \leq \|\omega\|$ with

$$\delta \leq \|y^\delta - F(x_\alpha^\delta)\| \leq c\delta \ , \tag{5.14}$$

and for a belonging minimizer holds $\|x_\alpha^\delta - x^\dagger\| = \mathcal{O}(\sqrt{\delta})$. A regularization parameter fulfilling (5.14) can be found by testing $\|y^\delta - F(x^\delta_{\alpha_k})\|$ for a sequence $\alpha_k = \alpha_0 q^k$ with appropriate chosen a_0 and $q < 1$, see [19].

Please note that if a solution fulfills smoothness condition from (II), then, for properly scaled $F'(x^\dagger)$, also a smoothness condition (5.8) holds. Thus, if (2.6) holds, all rules are conform with the requirements of our minimization algorithm. Combining all ingredients and picking a proper parameter rule we may provide the following algorithm which uses our iteration routine **TIREFU** (**TI**khonov **RE**placement **FU**ntional) for solving the nonlinear problem $F(x) = y$ with $\|y - y^\delta\| \le \delta$. The exact way for computing a solution $x_\alpha^\star$ goes as follows (applying III):

- Define a sequence $\{\alpha_n\}$ with $\alpha_n \stackrel{n\to\infty}{\longrightarrow} 0$, pick some r and set $x_0 = \bar{x}$ (initial value x_0 for the outer iteration)
- **while** $\|F(x_\alpha^\star) - y^\delta\| > r \cdot \delta$
  - $\alpha = \alpha_n$
  - pick an admissible C
  - $[x_\alpha^\star] =$ **TIREFU** (F, y^δ, C, α, x_0):
    $x_{k+1} = \arg\min_x \tilde{J}_\alpha(x, x_k)$ (solved by a **Fixed Point Iteration**)
    $x_\alpha^\star = \lim_{k\to\infty} x_k$
  - $x_0 = x_\alpha^\star$

  end

For this algorithm we may now formulate the following optimality result:

Theorem 15 *Assume that* (2.6) *holds. Then Tikhonov regularization with one of the parameter rules I-III, where the minimizers are computed by* **TIREFU** *, is an optimal regularization method.*

Since in any numerical realization we cannot treat infinite series (computing limits), we additionally have to incorporate a stopping rule. If $\Phi_\alpha(x, a)$ denotes the operator defined in (4.2), then the algorithm reads as follows:

- Define a sequence $\{\alpha_n\}$ with $\alpha_n \stackrel{n\to\infty}{\longrightarrow} 0$, pick some r, tolerances τ_1 and τ_2, set $x_\alpha^\star = \bar{x}$
- **while** $\|F(x_\alpha^\star) - y^\delta\| > r \cdot \delta$
  - $\alpha = \alpha_n$
  - pick an admissible C
  - $[x_\alpha^\star] =$ **TIREFU** (F, y^δ, C, α, x_0, τ_1, τ_2)
    $k = 0$
    while $\|x_{k+1} - x_k\| > \tau_1$
    $l = 0,\ x_{k,0} = x_k$
    Repeat

$l = l + 1$

$x_{k,l} = \Phi_\alpha(x_{k,l-1}, x_k)$

Until $\|x_{k,l} - x_{k,l+1}\| \le \tau_2$

$x_{k+1} = x_{k,l}$

$k = k + 1$

end

$x_\alpha^\star = x_k$

– $x_0 = x_\alpha^\star$

end

As we have pointed out, the strongest limitation of **TIREFU** is condition (2.6). However, this condition was only used once at the very end of our analysis, and we expect that it will be possible to weaken the condition. As Landweber iteration and Levenberg-Marquardt iteration work under a similar condition, we might compare **TIREFU** with these methods. Landweber iteration is known to be a slow method, and as we use fixed point methods, we do expect that **TIREFU** will be faster. Moreover, using our optimization routine with rule II, we obtain an optimal method for $\nu \in [1/2, 1]$. In contrast, to obtain convergence rates, Landweber requires an additional conditions $F'(x) = R_x F'(x^\dagger)$, where R_x is a family of bounded operators with $\|I - R_x\| \le K\|x - x^\dagger\|$. This condition is even more restrictive than (2.6). In addition, convergence rates are only available for $0 < \nu \le 1/2$. As for Levenberg-Marquardt, it is only known that the iteration is a regularization method under a condition slightly more restrictive as (2.6), and so far, nothing is known on convergence rates. Thus we might conclude that **TIREFU** works under less restrictive conditions.

6 Application of the proposed scheme

In this section, we want to apply the machinery developed in the previous sections. The aim is to demonstrate the capabilities and the performance of our algorithm in solving a challenging ill–posed problem in the context of medical imaging, which is Single Photon Emission Computerized Tomography (SPECT).

In SPECT, the patient gets a radiopharmaceutial, which is distributed through the whole body by the blood flow, and is finally enriched in some areas of interest. The task is to recover the distribution of the radiopharmaceutical (or, in short of the activity function f) from measurements of the radioactivity outside the body. In contrast to CT, where the measured intensity depends only on the intensity of the incoming X-ray and the density μ of the tissue along the path of the X-ray, the measurement for SPECT depend on the activity function f (which describes the distribution of the radiopharmaceutical) and the density μ of the tissue. The measured data y and the tuple (f, μ) are connected via the attenuated Radon Transform (ATRT),

$$y = R(f,\mu)(s,\omega) = \int_{\mathbb{R}} f(s\omega^\perp + t\omega) e^{-\int_t^\infty \mu(s\omega^\perp + r\omega)dr} dt \ , \tag{6.1}$$

where $s \in \mathbb{R}$ and $\omega \in S^1$. Usually both f and μ are unknown functions, and R is a nonlinear operator. In order to invert (6.1), two strategies can be used. Firstly, the density distribution

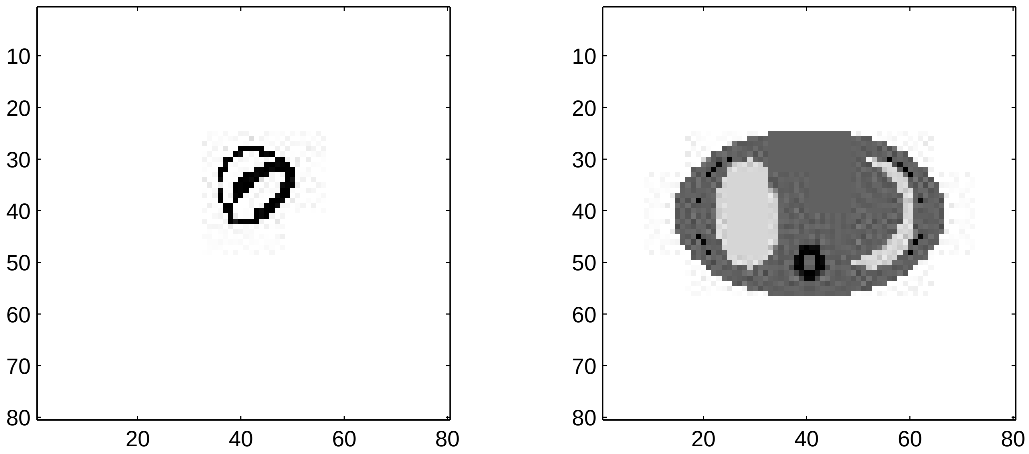


Figure 1: Activity function f_* (left) and attenuation function μ_* (right)

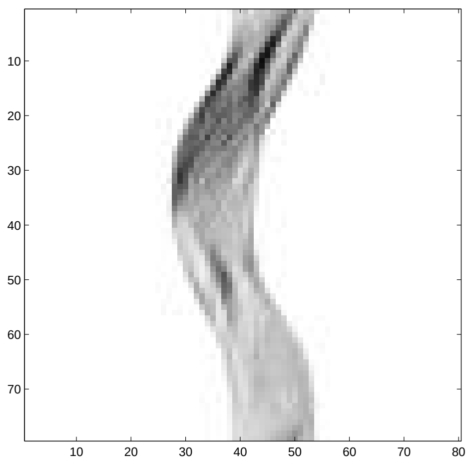


Figure 2: Generated data $g(s, \omega) = R(f_*, \mu_*)(s, \omega)$.

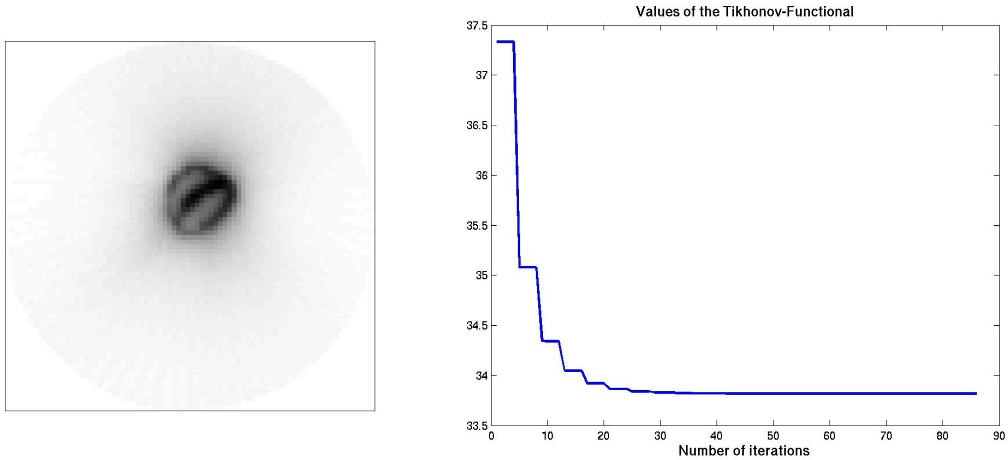
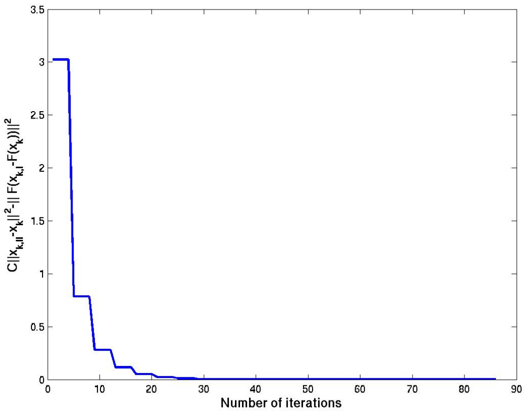


Figure 3: Minimizer f_α^δ of the Tikhonov functional for $\alpha = 3430$ (l) and values of the Tikhonov functional (r)

can be determined by the inversion of a additional CT scan (in most scanners, the CT data is gathered simultaneously). With this approach, one has to solve two linear problems, as the attenuated Radon transform is linear if μ is assumed to be known, and currently developed inversion formulas can be used [15]. However, attaching an X-ray source to a SPECT scanner makes them much more expensive. The scanning time for each patient might increase, which leads again to higher costs for each scan. Thus the second strategy, where the ATRT is treated as a nonlinear operator, seems to be promising. The drawbacks of this strategy are the non-uniqueness of the operator (which usually leads to a wrong reconstruction for the density function μ) and much higher computational costs for the inversion of the nonlinear operator. In the last decade, several ideas for solving the nonlinear problem (6.1) were discussed, see, e.g., [3, 14, 24, 25, 21]. Dicken [7] showed that Tikhonov regularization for nonlinear operators can be used for the reconstruction of the activity function. Methods for the computation of a minimizer of the Tikhonov functional were proposed in [18, 19, 20] and applied to SPECT. Here, we will only demonstrate that our method can be used for the computation of a minimizer. For the test computations, we would like to use the so called MCAT phantom [23], see Figure 1. The belonging sinogram data is shown in Figure 2.

In a first attempt, we want to compute the minimizer of the Tikhonov functional with regularization parameter $\alpha = 3430$. The data was contaminated with multiplicative Gaussian noise with relative error $\delta_{rel} = 5\%$ (here $\delta_{rel} = \|y^\delta - y\|/\|y\|$). The inner iteration was terminated if the relative distance of two consecutive iterates was less than $1e{-}6$, and the outer iteration was terminated if the relative distance between two consecutive outer iterates was less that $1e-5$. After only a few iterations, the value of the Tikhonov functional remains almost constant, see Figure 3. The values of the additive term $C\|x_{k,l} - x_k\| - \|F(x_{k,l}) - F(x_k)\|$, $x_k = (f_k, \mu_k)$ is shown in Figure 4. Clearly, the additive term converges fast to zero, and thus the values of the replacement functional and the Tikhonov functional are almost the same. Moreover, it turns out that we only need a few inner iterations to achieve the required accuracy, see Figure 4. This actually indicates that the whole iteration itself is quite fast. In a final test computation, we used Morozov's discrepancy principle to determine an appropriate regularization parameter



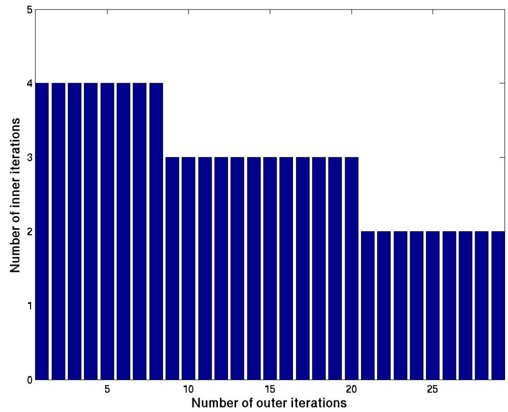


Figure 4: Plot of the additive term in the replacement functional (l) and number of inner iterations for each outer iteration (r)

(see (5.14)). For a sequence $\alpha_k = a_0 q^k$, $k = 0, 1, ...$, $a_0 = 1000$ and $q = 0.5$ we computed $x^\delta_{\alpha_k}$ by **TIREFU** , and picked the first minimizer $x^\delta_{\alpha_k}$ with (5.14) and $c = 2$. In our case, we had to compute 10 minimizing functions. The residual of the minimizer with $\alpha = 1.95$ was smaller then 2δ for the first time, and the reconstruction was stopped. Figure 5 shows the results.

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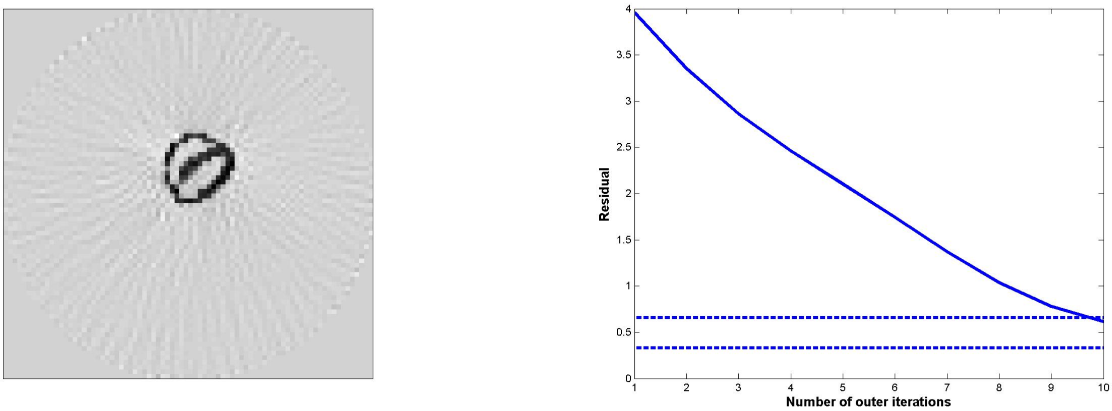

Figure 5: Left:Reconstruction according to Morozov's discrepancy principle. Right: Plot of the residual $\|y^\delta - R(f^\delta_{\alpha_k}, \mu^\delta_{\alpha_k})\|$. The dashed lines mark the region $[\delta, 2\delta]$.

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