Construction of Generalized Uncertainty Principles and Wavelets in Anisotropic Sobolev Spaces

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Abstract

This paper is concerned with the construction of generalized uncertainty relations. Starting with a two operator parabola ansatz we derive a new set of uncertainties by extending the parabola ansatz to quadratic forms. This setting can be applied to the computation of uncertainties where more than two operators are involved. For affine Weyl-Heisenberg groups some examples are explored.

Key Words: Affine Weyl-Heisenberg Groups, Minimal Uncertainty States, Uncertainty Principles, Wavelets

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1 Introduction

The main result of this paper is the construction of generalized uncertainty principles and the computation of related minimizing wavelets. The starting point is the uncertainty framework related to the Gabor and wavelet transform. It is known that the common thread between both transforms is group theory, see, e.g., [DM76, GMP85, GMP86, AAG91a, AAG91b, AAGM95, Fol98, AAG00]. In both cases one can associate with the underlying group representation so-called infinitesimal operators which directly lead to uncertainty relations. In the classical setting one can prove that for every non-commuting pair of such infinitesimal operators one can construct an uncertainty relation. The minimization leads to so-called minimal uncertainty states. These states have optimal 'group associated' localization properties. In the field of signal processing these kind of localization is of significant importance, see, e.g., [AMV99].

Whenever the group has two parameters only, e.g. time and frequency, there would be a canonical choice for the analyzing wavelet, namely the minimal uncertainty state. However, in many situations, e.g. the $ax + b$ group in two dimensions, it is known that there are more than one uncertainty, see [DM95], and thus there is no obvious choice of the analyzing wavelet. This becomes clear because no minimal uncertainty state is associated with all infinitesimal operators

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simultaneously. The goal of this paper is not to circumvent this deficiency completely but to improve the machinery by introducing a more general formulation of uncertainties, which allows to analyze situations where more than two infinitesimal operators are involved.

We shall mainly direct our attention to uncertainty principles arising in the context of mixed Gabor and wavelet analysis in a Sobolev space framework. Mixed Gabor–wavelet representations have been constructed during the last decade in order to improve the flexibility and to analyze the relations between Gabor and wavelet analysis, see, e.g., [Tor91, Tor92, KT93]. It was proposed to consider bigger groups containing both the affine and the Weyl-Heisenberg group. These groups clearly have at least three parameters (Frequency, Time, Dilation, ...). Consequently, it is not clear how to obtain the canonical coherent state associated with all involved infinitesimal operators. The application of the elaborated generalized uncertainty framework results not in the computation of an unique coherent state associated with all infinitesimal operators but it brightens the situation in a 'multi dimensional' operator setting, and, moreover, in the case where the parameter space is a homogeneous space.

The main idea of the multi operator construction is as follows. We start by recognizing that the classical two operator uncertainty is based on a certain parabola. In the presence of more than two operators, the generalization consists of incorporating quadratic forms instead of parabolas. This leads to matrix-valued commutator relations. Finally, one finds that in our setting every uncertainty principle can be expressed as a determinant. For two operators the determinant coincides with the discriminant which is the classical uncertainty formulation.

The paper is organized as follows:

1. In the second section, we review the needed ingredients of the theory on groups and group representations.
2. In the third section, we repeat the classical uncertainty framework and we present the generalized uncertainty setting which can be interpreted as a multi dimensional parabola ansatz.
3. In the fourth section, we introduce affine Weyl-Heisenberg groups, we derive the related admissibility conditions in anisotropic Sobolev spaces, and finally, we explore by two examples the applicability of the generalized uncertainty setting.

2 Group Theoretical Setting

Let G be separable Lie group with left or right Haar measure $d\mu$. A representation π of G in a Hilbert space $\mathcal{H}$ is a homomorphism between G and the group of bounded linear mappings $\mathcal{L}(\mathcal{H})$. We restrict ourselves to the setting of continuous unitary irreducible representations. Such a representation is called square integrable if π is irreducible and there exists a non-trivial vector $\psi \in \mathcal{H}$ such that

$$0 < \int_G |\langle \pi(g)\psi, \psi\rangle_{\mathcal{H}}|^2 d\mu(g) < \infty. \tag{2.1}$$

A vector ψ satisfying (2.1) is called admissible. In case where (2.1) does not exists one may restrict the integration to an adequate subset X. Often one uses $X = G/H$, where H is a closed subgroup of G. Because π acts on G and not on G/H it becomes necessary to embed G/H in G. This will be realized by the canonical fiber bundle structure of G, see [Mac76],

$$\Pi : G \to G/H.$$

Let σ be a section of such a fiber bundle. A representation acting on G/H is called a σ–modified representation and is defined by

$$\pi_\sigma = \pi \circ \sigma. \tag{2.2}$$

This leads to a slightly modified definition of admissibility, see [KT93]. A section σ is called admissible if there exists a bounded positive and invertible operator $\mathcal{A}$ with a bounded inverse and a vector ψ such that for all $f \in \mathcal{H}$

$$\int\limits_{G/H} |\langle \pi_\sigma(x)\psi, f\rangle_{\mathcal{H}}|^2 d\mu(x) = \langle f, \mathcal{A}f\rangle_{\mathcal{H}} \tag{2.3}$$

holds, where $d\mu$ is a quasi invariant measure on $L_2(G/H)$. A stronger formulation is given by the following definition. A section σ is called strictly admissible if there exists a positive constant K and a vector ψ such that for all $f \in \mathcal{H}$

$$\int\limits_{G/H} |\langle \pi_\sigma(x)\psi, f\rangle_{\mathcal{H}}|^2 d\mu(x) = K\|f\|^2_{\mathcal{H}} \tag{2.4}$$

holds. The general structure of σ was analyzed in [AAG91a, AAG91b]. By π and the definitions of admissibility one has well-defined integral transformations on $\mathcal{H}$. In dependence on the underlying Hilbert-space $\mathcal{H}$ and the special choice of G one has to state explicitly the conditions to specify the set of analyzing function.

3 Construction of Generalized Uncertainty Principles

The starting point is the underlying Lie group G and its representation π in some Hilbert space $\mathcal{H}$. The tangent space $\mathcal{G}$ to G at e consists of all elements X for which one can find a C^1 curve $g(\tau)$ that lies in G and satisfies $g(0) = e$ and $g'(0) = X$. $\mathcal{G}$ is called the Lie Algebra of G. By the representation π of G one has a corresponding representation π of its Lie Algebra $\mathcal{G}$, namely, the differential of the mapping π

$$\pi(X) = \left.\frac{d\pi(g(\tau))}{d\tau}\right|_{\tau=0} . \tag{3.1}$$

The operators $\pi(X)$ are called infinitesimal operators of the representation π of G. We associate with g_i the i-th parameter subgroup $g_i(\tau)$ and denote with $A(g_i)$ the related i-th infinitesimal operator. The main property for our purpose is that they lead to uncertainty relations. This result is well known, e.g., [Fol98]. In order to motivate our approach we include this result. To this end, we have to introduce some notations beforehand. Let A be an infinitesimal operator. The expectation of A with respect to a state $\psi \in \mathcal{D}(A)$ is defined by

$$\mu_\psi(A) := \frac{\langle A\psi, \psi\rangle}{\|\psi\|^2} \tag{3.2}$$

and the variance of A by

$$\Delta^2_\psi(A) := \mu_\psi((A - \mu_\psi(A))^2) = \mu_\psi(A^2) - \mu_\psi(A)^2 . \tag{3.3}$$

For self-adjoint and non-commuting operators A and B one can proof the following theorem.

Theorem 3.1 *Assume that A and B are non-commuting and self-adjoint operators and let the commutator be given by $[A, B] = iC$. Then for all $\psi \in \mathcal{D}([A, B])$ the following uncertainty relation*

$$\mu_\psi(C)^2 \leq 4\mu_\psi(A^2)\mu_\psi(B^2) \tag{3.4}$$

holds. One has equality in (3.4) *if and only if there exists a parameter $t \in \mathbb{R}$ with*

$$(A - itB)\psi = 0 \quad \text{or equivalently} \quad (A^2 + t^2B^2)\psi = -tC\psi \ . \tag{3.5}$$

Proof For all $t \in \mathbb{R}$ one has $(A - itB)^*(A - itB) = A^2 + tC + t^2B^2$. By $0 \leq \|(A - itB)\psi\|^2_{\mathcal{H}}$ the parabola $\mu_\psi(A^2) + t\mu_\psi(C) + t^2\mu_\psi(B^2)$ is real and nonnegative in t, and thus $\mu_\psi(C) - 4\mu_\psi(B^2)\mu_\psi(A^2) \leq 0$. Equality in (3.4) is equivalent to $(A - itB)\psi = 0$. ■

Considering instead of the infinitesimal operators A and B, respectively, the operators $A - \mu_\psi(A)$ and $B - \mu_\psi(B)$, respectively, the relation (3.4) can be expressed by means of variances

$$\mu_\psi(C)^2 \leq 4\Delta^2_\psi(A)\Delta^2_\psi(B).$$

Theorem 3.1 can be generalized in the following sense. Instead of two we incorporate r infinitesimal operators by switching to a vector-valued framework. The interplay of all these operators requires the introduction of the covariance of two operators A and B, which is defined by

$$\Delta_\psi(A, B) := \langle [A - \mu_\psi(A)]\psi, [B - \mu_\psi(B)]\psi \rangle \ . \tag{3.6}$$

Using this notation a generalization of Theorem 3.1 reads as follows.

Theorem 3.2 *Let a system of infinitesimal operators $\{A_l\}_{l=1,\ldots,r}$ be given. By*

$$A := (A_1, \ldots, A_r) \ , \ \ K := (-i[A_k, A_l])_{k,l=1,\ldots,r} \ \ \text{and} \ \ \Sigma := (\Delta(A_k, A_l))_{k,l=1,\ldots,r} \tag{3.7}$$

we define the vector of operators, the related commutator matrix and the related matrix of co-variances. Assume that we have two linear combinations of the form $B = A \cdot \gamma'$ and $C = A \cdot \alpha'$, where $\gamma, \alpha \in \mathbb{R}^r$, such that $\gamma K \alpha' \neq 0$. Then, for all $\psi \in \mathcal{D}([B, C])$ the following uncertainty principle

$$\mu_\psi(\gamma K \alpha')^2 \leq 4\gamma\Sigma\gamma'\alpha\Sigma\alpha' \tag{3.8}$$

holds. One has equality in (3.8) *if and only if there exists a parameter $t \in \mathbb{R}$ with*

$$(B - itC)\psi = 0 \quad \text{or equivalently} \quad (C^2 + t^2B^2)\psi = -t\gamma K\alpha'\psi \ . \tag{3.9}$$

Proof Just as in the proof of Theorem 3.1 we compute $(B - itC)^*(B - itC)$. Thus, we have to compute the $\left[\sum_j \gamma_j A_j, \sum_l \alpha_l A_l\right]$. By $[A_j, A_l] = -[A_l, A_j]$ it follows that

$$\begin{aligned}
\left[\sum_j \gamma_j A_j, \sum_l \alpha_l A_l\right] &= \sum_j \sum_l \gamma_j \alpha_l [A_j, A_l] = \sum_{j,l;j\neq l} \gamma_j \alpha_l [A_j, A_l] \\
&= \ (\gamma_1\alpha_2 - \gamma_2\alpha_1)[A_1, A_2] + (\gamma_1\alpha_3 - \gamma_3\alpha_1)[A_1, A_3] + \ldots \\
&\qquad (\gamma_2\alpha_3 - \gamma_3\alpha_2)[A_2, A_3] + \ldots
\end{aligned}$$

$$= \quad (\gamma_1, \ldots, \gamma_r) \begin{pmatrix} 0 & [A_1, A_2] & \ldots & [A_1, A_r] \\ -[A_1, A_2] & 0 & & \vdots \\ \vdots & & \ddots & [A_{r-1}, A_r] \\ -[A_1, A_r] & \cdots & -[A_{r-1}, A_r] & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}$$

$$= \quad i\gamma K \alpha' \ .$$

Hence, we have $(B - itC)^*(B - itC) = B^2 + t\gamma K\alpha' + t^2 C^2$. Consequently, for all $\psi \in \mathcal{D}([B, C])$ the equation

$$\|(B - itC)\psi\|^2_{\mathcal{H}} = \mu_\psi(B^2) + t\mu_\psi(\gamma K\alpha') + t^2\mu_\psi(C^2) \tag{3.10}$$

holds. Non-negativity of (3.10) implies

$$\mu_\psi(\gamma K\alpha')^2 \leq 4\mu_\psi(B^2)\mu_\psi(C^2) \ . \tag{3.11}$$

One has equality in (3.11) if and only if there exists a parameter $t \in \mathbb{R}$ with $(B - itC)\psi = 0$ or equivalently $(B^2 + t^2C^2)\psi = -t\gamma K\alpha'\psi$. To rewrite (3.8) we use that $B = A\gamma'$ and $C = A\alpha'$. Then, it follows immediately that

$$\Delta^2_f(A\gamma') = \Delta_f(A\gamma', A\gamma') = \left\langle A\gamma' f, A\gamma' f\right\rangle = \gamma\Sigma\gamma' \quad , \tag{3.12}$$

and similarly one has $\Delta^2_f(A\alpha') = \alpha\Sigma\alpha'$. ∎

The next corollary shows an obvious decomposition in pure variances and mixed terms of relation (3.8).

Corollary 3.1 *The uncertainty relation* (3.8) *can be expressed as follows*

$$\frac{1}{4}\mu_\psi(\gamma K\alpha')^2 \leq \gamma V\gamma'\alpha V\alpha' + M(\alpha, \gamma) \ ,$$

where $V = \operatorname{diag}\left(\Delta^2_f(A_1), \ldots, \Delta^2_f(A_r)\right)$ *and* $M(\alpha, \gamma) = \gamma(\Sigma - V)\gamma'\alpha V\alpha' + \gamma\Sigma\gamma'\alpha(\Sigma - V)\alpha'$.

In order to interpret the new obtained uncertainties in a different way we formulate them in terms of parabolas. To this end, we consider as before linear combinations B and C. We may now establish the following representation of r-operator uncertainty relations.

Proposition 3.1 *Let the infinitesimal operators* $\{A_l\}_{l=1,\ldots,r}$ *be given, and define a matrix* F *by*

$$F := \left(\begin{array}{cc|cc} A_1^2 & (A_iA_j + A_jA_i)/2 & 0 & i[A_i, A_j]/2 \\ & \ddots & & \ddots \\ (A_iA_j + A_jA_i)/2 & A_r^2 & i[A_j, A_i]/2 & 0 \\ \hline 0 & i[A_i, A_j]/2 & A_1^2 & (A_iA_j + A_jA_i)/2 \\ & \ddots & & \ddots \\ i[A_j, A_i]/2 & 0 & (A_iA_j + A_jA_i)/2 & A_r^2 \end{array}\right) .$$

Let the matrix $\mu_\psi(F)$ *denote the component-wise expectation of* F. *Then for some* $\psi \in \mathcal{D}([B, C])$ *the quadratic form* $\mu_\psi(F)$ *is non-negative and the determinant of* $\mu_\psi(F)$ *establishes an uncertainty relation*

$$\det\left(\mu_\psi(F)\right) \geq 0 \ .$$

Proof At first, by setting $\eta = t\alpha$ we observe that

$$(B - itC)^*(B - itC) = (A\gamma' + iA\eta')(A\gamma' - iA\eta') = (\gamma, \eta)F(\gamma, \eta)' \quad .$$

Hence, for all $(\gamma, \eta) \in \mathbb{R}^{2r}$ it follows that

$$0 \le \|(B - itC)\psi\|^2 = (\gamma, \eta)\mu_\psi(F)(\gamma, \eta)' \ ,$$

where $\mu_\psi(F)$ denotes the component-wise expectation of F, i.e.

$$F = \left(\begin{array}{cc|cc} \Delta^2_\psi(A_1) & \Re\Delta_\psi(A_i, A_j) & 0 & \mu_\psi(i[A_i, A_j]/2) \\ & \ddots & & \cdots \\ \Re\Delta_\psi(A_i, A_j) & \Delta^2_\psi(A_r) & \mu_\psi(i[A_j, A_i]/2) & 0 \\ \hline 0 & \mu_\psi(i[A_i, A_j]/2) & \Delta^2_\psi(A_1) & \Re\Delta_\psi(A_i, A_j) \\ & \ddots & & \cdots \\ \mu_\psi(i[A_j, A_i]/2) & 0 & \Re\Delta_\psi(A_i, A_j) & \Delta^2_\psi(A_r) \end{array} \right) .$$

Consequently, $\mu_\psi(F)$ is non-negative, and an uncertainty can be written as the determinant. ■

We remark that for the two operator case the determinant is just the discriminant. The uncertainty formulation (3.8) can be obtained by observing that F can be expressed by means of K and Σ.

4 Uncertainty Principles for Affine Weyl-Heisenberg Groups

In this section, we relate the established uncertainty framework to so-called affine Weyl-Heisenberg groups. These groups appear in the context of Fourier and wavelet analysis and arise by mixing the affine and the Weyl-Heisenberg group. For both groups uncertainty relations and minimizing states are well known, e.g. [DM95, Fol98]. However, so far there exist only uncertainty formulations for pairs of infinitesimal operators. In the following we apply the above established machinery to show how one can create uncertainties for more than two operators. This does not imply that we can avoid the presence of several uncertainties. The advantage of the new machinery is to extend the flexibility in formulating uncertainties. Here we focus on the special case where the underlying group is a affine Weyl-Heisenberg group and where the function spaces are (anisotropic) Sobolev spaces.

4.1 Affine Weyl-Heisenberg Groups

We start by briefly introducing the groups under consideration. The affine group is given by

$$\mathbf{G}_{aff} = (\{(q, a, R) : \ q \in \mathbb{R}^n, \ a \in \mathbb{R}_+, \ R \in SO(n)\}, \circ),$$

with group law $(q, a, R) \circ (q', a', R') = (q + aRq', aa', RR')$. It is well known, e.g. [SD80], that the right and left Haar measures on $\mathbf{G}_{aff}$ are given by $d\mu_R(q, a, R) = a^{-1}dadqdm(R)$ and $d\mu_L(q, a, R) = a^{-(n+1)}dadqdm(R)$, where $dm(R)$ denotes the invariant measure on $SO(n)$. Beside the affine group the next group of interest is the reduced Weyl-Heisenberg group given by

$$\mathbf{G}_{WH} = \left(\{(q, p, \varphi) : \ q \in \mathbb{R}^n, \ p \in \mathbb{R}^n, \ \varphi \in S^1, \circ\}\right) ,$$

where the group law is given by $(q, p, \varphi) \circ (q', p', \varphi') = (q + q', p + p', \varphi + \varphi' + p \cdot q' \ mod\ 2\pi)$. It is also a known fact, e.g. [SD80], that $\mathbf{G}_{WH}$ is unimodular with invariant measure $d\nu(g) = dqdpd\varphi$.

We are still interested in mixing these two groups. This topic was already considered in, e.g., [Tor91, KT93, Tor94, AAGM95, AAG91a, AAG91b]. The mixing can be realized by direct or semi-direct group products. The so-called direct affine Weyl-Heisenberg group, denoted by $\mathbf{G}^d_{aWH}$, is defined as the direct product of $\mathbf{G}_{WH}$ and $\mathbf{G}_{aff}$ an is equipped with the component-wise group law

$$(q,p,\varphi;b,a,R)\circ(q',p',\varphi';b',a',R') =$$
$$(q+q',p+p',\varphi+\varphi'+pq' \ mod\ 2\pi; b+aRb',aa',RR').$$

There are different ways to introduce a suitable group representation. For our purpose we follow a common technique, see [VK91], and construct a suitable representation by a direct product, i.e. if π_{WH} and π_{aff} are given irreducible representations of $\mathbf{G}_{WH}$ and $\mathbf{G}_{aff}$ respectively, then

$$\pi^{\mathsf{d}}(g,h) = \pi_{WH}(g)\otimes\pi_{aff}(h), \tag{4.1}$$

with $g\in\mathbf{G}_{WH},\ h\in\mathbf{G}_{aff}$ is a irreducible representation of $\mathbf{G}_{WH}\times\mathbf{G}_{aff}$ (the superscript d indicates the direct product). Moreover, the related left (right) Haar measure is the product measure $\nu\otimes\mu_L$ ($\nu\otimes\mu_R$), see [VK91, SD80]. The second alternative is the semi-direct product of $\mathbf{G}_{aff}$ and $\mathbf{G}_{WH}$. Here we mainly follow the approach of [KT93], i.e. we introduce the semi-direct affine Weyl-Heisenberg group $\mathbf{G}_{aWH}$ as the semi-direct product of $\mathbf{G}_{WH}$ and $\mathbf{G}_{aff}$ equipped with the group law

$$(q,p,a,R,\varphi)\circ(q',p',a',R',\varphi')$$
$$=(q+aRq',p+a^{-1}Rp',aa',RR',\varphi+\varphi'+p(aRq'))\ .$$

For our purposes we choose the irreducible Stone-von-Neumann representation

$$\pi^{\mathsf{s}}(q,p,a,R,0)f(x) = a^{-n/2}e^{ip(x-q))}f(R'(x-q)/a). \tag{4.2}$$

One easily checks that $\mathbf{G}_{aWH}$ is unimodular with Haar measure $d\mu(q,p,a,R,\varphi) = dqdpdadm(R)d\varphi/a$.

4.2 Sobolev Spaces and Admissibility

Let $H^s(\mathbb{R}^n)$ denote the common Sobolev space. In order to establish a suitable group representation we introduce the operator Λ_s by

$$\Lambda_s\ :\ S'(\mathbb{R}^n)\to S'(\mathbb{R}^n),\quad \text{where}\quad (\Lambda_s f)^\wedge(\omega) = (1+\|\omega\|^2)^{s/2}\hat{f}(\omega). \tag{4.3}$$

Consequently, a representation of some group G in $H^s(\mathbb{R}^n)$ can now be defined by its representation π in $L_2(\mathbb{R}^n)$

$$\pi_s(g) := \Lambda_{-s}\pi(g)\Lambda_s\ :\ \ H^s(\mathbb{R}^n)\to H^s(\mathbb{R}^n). \tag{4.4}$$

To apply (4.3) in definition (4.4) we have to move from π to an equivalent representation $\hat{\pi}$ in phase space via

$$\pi = \mathcal{F}^{-1}\hat{\pi}\mathcal{F}\quad \text{if and only if}\quad \mathcal{F}\pi = \hat{\pi}\mathcal{F}\ . \tag{4.5}$$

Let us start by the direct group product $\mathbf{G}_{WH}\times\mathbf{G}_{aff}$. To this end, we firstly identify a suitable underlying (anisotropic) Sobelev space. We define the Hilbert space

$$H^{s_1,s_2}_{mix}(\mathbb{R}^m\times\mathbb{R}^n) := \{u\in L_2(\mathbb{R}^n\times\mathbb{R}^m)\ :\ \ \|u\|_{s_1,s_2}<\infty\}\quad , \tag{4.6}$$

which is equipped with the norm

$$\|u\|^2_{mix} := \int\limits_{\mathbb{R}^m\times\mathbb{R}^n} |(1+\|k_1\|^2)^{s_1/2}(1+\|k_2\|^2)^{s_2/2}\mathcal{F}u(k_1,k_2)|^2dk_1dk_2\ . \tag{4.7}$$

One can show that H^{s_1,s_2}_{mix} is a tensor product Hilbert space, i.e. for any $s_1, s_2 \geq 0$ and $n, m \in \mathbb{N}$ we have $H^{s_1,s_2}_{mix}(\mathbb{R}^m \times \mathbb{R}^n) = H^{s_1}(\mathbb{R}^m) \otimes_l H^{s_2}(\mathbb{R}^n)$, see appendix for the definition of $\otimes_l$ and e.g. [Hoc98]. By (4.1) and (4.5) we obtain the following representation of $\mathbf{G}^d_{aWH}$ in phase space

$$\begin{aligned} \hat{\pi}^{\mathsf{d}}_{s_1,s_2}(g,h)(\hat{f})(\omega,\eta) \quad := \quad & (1+\|\omega\|^2)^{-s_1/2}(1+\|\eta\|^2)^{-s_2/2}e^{i(\varphi-q'\omega)}a^{n/2}e^{-ib'\eta} \quad \times \\ & (1+\|\omega+p\|^2)^{s_1/2}(1+\|aR\eta\|^2)^{s_2/2}\hat{f}(\omega+p,aR\eta) \ , \end{aligned}$$

i.e. the Fourier multiplier Λ_s in Definition (4.4) is given by

$$\left(\Lambda^{mix}_{s_1,s_2} f\right)^{\wedge}(k_1,k_2) := (1+\|k_1\|^2)^{s_1/2}(1+\|k_2\|^2)^{s_2/2}\hat{f}(k_1,k_2) \ .$$

Lemma 4.1 *The representation $\pi^{\mathsf{d}}_{s_1,s_2}$ of $\mathbf{G}^d_{aWH}$ in $H^{s_1}(\mathbb{R}^m) \otimes_l H^{s_2}(\mathbb{R}^n)$ is homomorphic, continuous, unitary and square integrable.*

Remark 4.1 In principle, the result of Lemma 4.1 holds also for other reasonable Fourier multipliers, see [Tri78, ST87], e.g.

$$\Lambda^A_{s_1,s_2} = (1+\|k_1\|^2)^{s_1/2} + (1+\|k_2\|^2)^{s_1/2} \text{ or } \Lambda^T_{s_1,s_2} = (1+\|(k_1,k_2)\|^2)^{s_1/2}(1+\|k_1\|^2).$$

The related Sobolev spaces satisfy for $0 \leq s_1, s_2 \leq \mathbb{R}$ and $m, n \in \mathbb{N}$ and $\bar{s} = (s_1, s_2)$ and $\tilde{s} = (s_1, \max(s_1, s_2))$ the relation $H^{2\tilde{s}}_A \subseteq H^{\bar{s}}_T \subseteq H^{\bar{s}}_{mix}$.

We summarize the results in the following

Proposition 4.1 *Let $\Lambda^*_{\bar{s}}$ denote a Fourier multiplier in the sense of Definition* (4.4)*, then a functions $f \in H^{\bar{s}}_*$ is admissible if and only if*

$$0 < (2\pi)^{m+n+1} vol(SO(n-1))\|f\|^2_* \int_{\mathbb{R}^{m+n}} \frac{|(\Lambda^*_{\bar{s}} f)^\wedge(k_1,k_2)|^2}{\|k_2\|^n} d^m k_1 d^n k_2 < \infty \ . \tag{4.8}$$

The operator $\mathcal{A}$ in condition (2.3) *is given by*

$$\mathcal{A} = Id \cdot C \int_{\mathbb{R}^{m+n}} \frac{|(\Lambda^*_{\bar{s}} f)^\wedge(k_1,k_2)|^2}{\|k_2\|^n} d^m k_1 d^n k_2 \tag{4.9}$$

and hence, by (2.4) *the related Gabor-wavelet-transform is an isometry.*

The situation is completely different for the semi-direct group product $\mathbf{G}_{aWH}$ and related representations. The first remarkable fact is stated in the following

Lemma 4.2 *The representation* (4.2) *of the semi-direct product $\mathbf{G}_{aWH}$ in $L_2(\mathbb{R}^n)$ is not square integrable.*

The situation does not change if we switch to π^{s}_s, i.e. $\pi^{\mathsf{s}}_s = \Lambda^*_{-s}\pi^{\mathsf{s}}\Lambda^*_s$ is also not square integrable in $H^{\bar{s}}_*$. To overcome this deficiency we follow [KT93] and proceed as described in Section 2. This means that we have to restrict the integration to suitable cosets X. The representation π^{s}_s acts on $\mathbf{G}_{aWH}$ but not on X, hence one has to find an embedding (section) σ such that $\pi^{\mathsf{s}}_\sigma = \pi^{\mathsf{s}} \circ \sigma$ becomes a representation of X. There are several alternatives to choose X, see again for details [KT93]. Here we limit the analysis to one special instance

$$X = \Gamma \setminus \mathbf{G}_{aWH} \quad \text{with} \quad \Gamma = \{(0,p,1,1,\varphi) \in \mathbf{G}_{aWH}\} \ . \tag{4.10}$$

The related admissibility condition is summarized in the following proposition.

Proposition 4.2 *Let* $f \in H^{\bar{s}}(\mathbb{R}^n)$ *and* $\sigma(q,a,R) = (q, \beta(a,R), a, R, 0)$*, where* $\beta : X \to \Gamma$ *is a piecewise differentiable mapping and let the representation* $\pi^{\mathsf{s}}_{\sigma,s}$ *be given by*

$$\Lambda^*_{-s}(\pi^{\mathsf{s}} \circ \sigma)\Lambda^*_s.$$

Then, the section σ *is strictly admissible for a constant co-vector* v *if*

$$0 < \int_{\mathbb{R}^n} \frac{|(\Lambda^*_s f)^\wedge(k)|^2}{|1 - \langle v, k\rangle\ |} dk < \infty \ . \tag{4.11}$$

This result can be deduced without any difficulties by reducing the proof to L_2-functions and following the lines in [KT93]. We remark that a possible structure of β is given by

$$\beta(a,R) = \frac{a^{-1}Rv'}{\|v\|^2} + \varrho(S) \ , \tag{4.12}$$

where $S \in SO(n-1)$, v is a $SO(n-1)$-invariant co-vector and ϱ is smooth enough on $SO(n-1)$.

4.3 Generalized Uncertainties and Minimizing Wavelets

Before computing some uncertainty principles and corresponding minimal uncertainty states we note that the modified representations π^{d}_s, $\pi^{\mathsf{s}}_{s,\sigma}$ induce infinitesimal operators $A(g_i)$ acting on $H^{\bar{s}}_*(\mathbb{R}^n)$. For convenience we simply denote $\pi_s = \Lambda_{-s}\pi\Lambda_s$ (the incorporation of the superscripts d and s and the embedding σ can be treated in the same manner).

At first, we explore an example for $\mathbf{G}^d_{aWH}$. Let the Hilbert space under consideration be $H^{\bar{s}}_{mix}(\mathbb{R}^2)$. In this case we have to incorporate the following infinitesimal operators $A(q)$, $A(p)$, $A(a)$, $A(b)$, $A(\varphi)$. Neglecting the toral operator $A(\varphi)$, we may consider the vector $A = (A(q), A(p), A(a), A(b))$. Now we choose two appropriate vectors γ and α, namely $\gamma = (0,1,1,0)$ and $\alpha = (1,0,0,1)$. Then we obtain the following commutator matrix

$$K \quad = \quad \begin{pmatrix} 0 & A(\varphi) & 0 & 0 \\ -A(\varphi) & 0 & 0 & 0 \\ 0 & 0 & 0 & A(b) \\ 0 & 0 & -A(b) & 0 \end{pmatrix} .$$

The chosen linear combinations $A\gamma'$ and $A\alpha'$ satisfy $\gamma K \alpha' = (0, -A(\varphi), A(b), 0) \cdot (0,1,1,0)' = -A(\varphi) + A(b) \neq 0$ (this represents the generalized commutator). In order to determine the uncertainty structure we have to compute V and Σ. Calculus shows that we obtain the following uncertainty relation

$$\begin{aligned} \mu_{\bar{s},f}(-A(\varphi) + A(b))^2 \quad \leq \quad & 4\left[\Delta^2_{\bar{s},f}(A(p)) + \Delta^2_{\bar{s},f}(A(a))\right]\left[\Delta^2_{\bar{s},f}(A(q)) + \Delta^2_{\bar{s},f}(A(b))\right] \\ & + \left[\Delta^2_{\bar{s},f}(A(q)) + \Delta^2_{\bar{s},f}(A(b))\right]\left[\Delta_{\bar{s},f}(A(p),A(a)) + \overline{\Delta_{\bar{s},f}(A(p),A(a))}\right] \\ & + \left[\Delta^2_{\bar{s},f}(A(p)) + \Delta^2_{\bar{s},f}(A(a))\right]\left[\Delta_{\bar{s},f}(A(q),A(b)) + \overline{\Delta_{\bar{s},f}(A(q),A(b))}\right] \ . \end{aligned}$$

The minimizing wavelet ψ is given in phase space as a solution of the following partial differential equation

$$(A(a) + A(p) - \mu_1)\,\hat{\psi} = it\,(A(b) + A(q) - \mu_2)\,\hat{\psi} \quad . \tag{4.13}$$

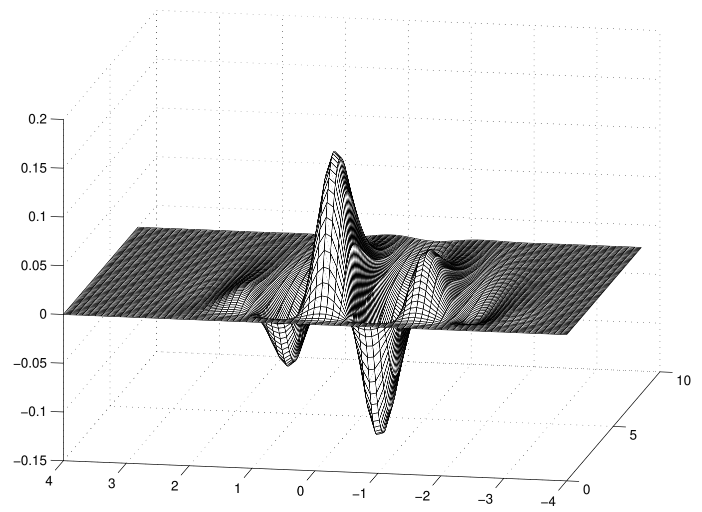

Figure 1: Wavelet function based on $\mathbf{G}^d_{aWH}$ for $s_1 = 1$, $s_2 = 5$, $t = -1$, $\mu_1 = -4$ and $\mu_2 = 6$.

A function ψ satisfying (4.13) is given by

$$\hat{\psi}(\omega,\eta) = \begin{cases} \eta^{1+s_2} e^{t\omega^2/2-\mu_2 t\omega-\omega/2-i\mu_1\omega+\eta t-s_2\omega-\omega} \left(1+\omega^2\right)^{-s_1/2} \left(1+\eta^2\right)^{-s_2/2} \ , \text{ for } (\omega,\eta) \in \mathbb{R}\times\mathbb{R}_+ \\ 0 \ , \text{ for } (\omega,\eta) \in \mathbb{R}\times\mathbb{R}_- \quad , \end{cases}$$

where the parameters t, μ_1 and μ_2 must be chosen such that ψ fulfills the admissibility condition (4.8). Figure 1 shows the minimizing wavelet for a special admissible choice of parameters.

We finish this section with constructing uncertainties related to $\mathbf{G}_{aWH}$. We proceed in two steps. At first, based on a homogeneous space X, we directly compute the uncertainty structure. Then, we observe that the obtained structure can also be created by applying Theorem 3.2 for special choices of γ and α. For convenience, we limit ourselves as in Section 4.2 to the homogeneous space X induced by the subgroup Γ given by (4.10) and to β given by (4.12). To keep notations at a reasonable level we choose $\mathcal{H} = H^s(\mathbb{R}^2)$. The instance shows how the infinitesimal operators dependent on σ and β.

Proposition 4.3 *Let $\mathbf{G}_{aWH}$ be the underlying group and let $\pi^{\mathsf{s}}_{s,\sigma} = \Lambda_{-s}(\pi^{\mathsf{s}}\circ\sigma)\Lambda_s$ be the square integrable representation of X in $H^s(\mathbb{R}^2)$. If the embedding function is given by*

$$\beta(a,R) = a^{-1}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{4.14}$$

where $a \in \mathbb{R}_+$ and $\theta \in [0,2\pi)$, then the infinitesimal operators are given by

$$A^\sigma(a) = A(a) + A(p_2) \ , \quad A^\sigma(q_j) = A(q_j) \text{ and } A^\sigma(\theta) = A(\theta) + A(p_1) \ . \tag{4.15}$$

Moreover, the infinitesimal operators satisfy the following commutator relations

$$[A^\sigma(a), A^\sigma(q_j)] = iA(q_j) + i\delta_{2j}A(\varphi) \ , \quad [A^\sigma(a), A^\sigma(\theta)] = 2iA(p_1) \ ,$$

$$[A^\sigma(\theta), A^\sigma(q_1)] = iA(q_2) + iA(\varphi) \ , \quad [A^\sigma(\theta), A^\sigma(q_2)] = -iA(q_1) \text{ and } [A^\sigma(q_1), A^\sigma(q_2)] = 0 \ . \tag{4.16}$$

Consequently, for the homogeneous space X there exist at most five uncertainty principles.

Proof We choose $n = 2$, $v = (0,1)$, $S \in SO(n-1)$, $\tau = -1$ and finally $\varphi(S) = (0,\tau)' = (0,-1)'$. Hence, the structure of (4.12) is satisfied

$$\beta(a,R) = a^{-1}(-\sin\theta,\cos\theta)' - (0,1)' = \frac{a^{-1}Rv'}{\|v\|^2} + \varphi(S) \ .$$

Using this embedding and the shorthand notation $\lambda(\omega) = (1+\|\omega\|^2)^{-s/2}$, we obtain the following infinitesimal operators

$$\begin{aligned}
\tilde{A}^\sigma(a)\hat{\psi}(\omega) &= \frac{\partial}{\partial a}\hat{\pi}_{s,\sigma}\hat{\psi}(\omega)\Big|_e = \lambda(\omega)(1+\|\omega\|^2)^{s/2}\hat{\psi}(\omega) + \\
&\quad \lambda(\omega)s(\|\omega\|^2+\omega_2)(1+\|\omega\|^2)^{s/2-1}\hat{\psi}(\omega) + \lambda(\omega)(1+\|\omega\|^2)^{s/2}\nabla\hat{\psi}\cdot\begin{pmatrix}\omega_1\\ \omega_2+1\end{pmatrix} \\
&= \hat{\psi}(\omega) + \frac{s\|\omega\|^2}{1+\|\omega\|^2}\hat{\psi}(\omega) + \nabla\hat{\psi}\cdot\omega + \frac{s\omega_2}{1+\|\omega\|^2}\hat{\psi}(\omega) + \frac{\partial}{\partial\omega_2}\hat{\psi}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{A}^\sigma(\theta)\hat{\psi}(\omega) &= \frac{\partial}{\partial\theta}\hat{\pi}_{s,\sigma}\hat{\psi}(\omega)\Big|_e = \lambda(\omega)s\omega_1(1+\|\omega\|^2)^{s/2-1}\hat{\psi}(\omega) + \\
&\quad \lambda(\omega)(1+\|\omega\|^2)^{s/2}\nabla\hat{\psi}\cdot\begin{pmatrix}\omega_2+1\\ -\omega_1\end{pmatrix} \\
&= \frac{s\omega_1}{1+\|\omega\|^2}\hat{\psi}(\omega) + \nabla\hat{\psi}\cdot\begin{pmatrix}\omega_2\\ -\omega_1\end{pmatrix} + \frac{\partial}{\partial\omega_1}\hat{\psi}(\omega) \ .
\end{aligned}$$

To make them self-adjoint we modify them by

$$i\tilde{A}^\sigma(a) =: A^\sigma(a) = A(a) + A(q_1) \ \text{ and } \ i\tilde{A}^\sigma(\theta) =: A^\sigma(\theta) = A(\theta) + A(p_1) \ .$$

This shows (4.15). Furthermore, we have

$$\tilde{A}^\sigma(q_j)\hat{\psi}(\omega) = \frac{\partial}{\partial q_j}\hat{\pi}_{s,\sigma}\hat{\psi}(\omega)\Big|_e = -i\omega_j\hat{\psi}(\omega\ ) \text{ and } \ i\tilde{A}^\sigma(q_j) =: A^\sigma(q_j) = A(q_j) \ .$$

Hence, the relations (4.16) follow easily

$$[A^\sigma_a, A^\sigma(q_j)] = [A(a), A(q_j)] + [A(p_2), A(q_j)] = iA(q_j) + i\delta_{j2}A(\varphi) \ ,$$

$$[A^\sigma(q_1), A^\sigma(q_2)] = [A(q_1), A(q_2)] = 0 \ ,$$

$$[A^\sigma_a, A^\sigma(\theta)] = [A(p_2), A(\theta)] + [A(a), A(p_1)] = 2iA(p_1) \ ,$$

$$[A^\sigma(\theta), A^\sigma(q_1)] = [A(\theta), A(q_1)] + [A(p_1), A(q_1)] = iA(q_2) + iA(\varphi) \quad \text{and}$$

$$[A^\sigma(\theta), A^\sigma(q_2)] = [A(\theta), A(q_2)] + [A(p_1), A(q_2)] = -iA(q_1) \ .$$

■

We note that the influence of the embedding function β is essentially reflected by the additive structure of the infinitesimal operators $A^\sigma(a)$ and $A^\sigma(\theta)$. In the following we limit the analysis to the uncertainties with respect to $A^\sigma(\theta)$ and $A^\sigma(q_l)$, for $l = 1,2$.

Proposition 4.4 *Let X be the homogeneous space given by* (4.10) *and let* $\psi \in H^s$ *with* $\|\psi\|_s = 1$. *Then the uncertainties with respect to* $A^\sigma(\theta)$ *and* $A^\sigma(q_l)$, $l = 1, 2$, *are given by*

$$\begin{aligned}\frac{1}{4}C_l \quad &\leq \quad \Delta^2_{s,\psi}(A(\theta))\Delta^2_{s,\psi}(A(q_l)) + \Delta^2_{s,\psi}(A(p_1))\Delta^2_{s,\psi}(A(q_l)) + \\ &\quad 2\Delta^2_{s,\psi}(A(q_l))\left[\Re\mu_{s,\psi}(A(\theta)A(p_1)) - \mu_{s,\psi}(A(\theta))\mu_{s,\psi}(A(p_1))\right] \ , \end{aligned} \tag{4.17}$$

with lower bounds

$$C_1 = \mu^2_{s,\psi}(A(q_2) + A(\varphi)) \quad and \quad C_2 = \mu^2_{s,\psi}(A(q_1)). \tag{4.18}$$

Proof This result can be deduced in the following way. Assume that A, B and C are some given self-adjoint operators. At first, we observe that

$$\begin{aligned}\Delta^2_{s,\psi}(A+B) \quad &= \quad \Delta^2_{s,\psi}(A) + \Delta^2_{s,\psi}(A) + \langle (AB+BA)\psi, \psi\rangle_s - 2\langle A\psi,\psi\rangle_s \langle B\psi,\psi\rangle_s \\ &= \quad \Delta^2_{s,\psi}(A) + \Delta^2_{s,\psi}(A) + 2\Re\langle B\psi, A\psi\rangle_s - 2\langle A\psi,\psi\rangle_s \langle B\psi,\psi\rangle_s \ .\end{aligned}$$

A Multiplication with $\Delta^2_{s,\psi}(C)$ and inserting the lower bounds proves the assertion. ■

By the definition of the covariance we may specify (4.17) in Proposition 4.4

$$\begin{aligned}\frac{1}{4}C_l \quad &\leq \quad \Delta^2_{s,\psi}(A(\theta))\Delta^2_{s,\psi}(A(q_l)) + \Delta^2_{s,\psi}(A(p_1))\Delta^2_{s,\psi}(A(q_l)) + \\ &\quad \Delta^2_{s,\psi}(A(q_l))\Delta_{s,\psi}(A(\theta), A(p_1)) + \Delta^2_{s,\psi}(A(q_l))\Delta_{s,\psi}(A(p_1), A(\theta)) \ .\end{aligned} \tag{4.19}$$

After deriving the uncertainties in some direct way, we show how we can obtain the same uncertainties simply by picking some special vectors γ, α_1, and α_2 (these vectors must be clearly related to β) followed by the application of Theorem 3.2. We again start with the same set of infinitesimal operators, i.e. $A = (A(p_1), A(p_2), A(q_1), A(q_2), A(\theta), A(a))$. Moreover, we choose $\gamma = (1, 0, 0, 0, 1, 0)$, $\alpha_1 = (0, 0, -1, 0, 0, 0)$, and $\alpha_2 = (0, 0, 0, -1, 0, 0)$. Then the commutator matrix reads as

$$K = \begin{pmatrix} 0 & 0 & -A(\varphi) & 0 & A(p_2) & -A(p_1) \\ 0 & 0 & 0 & -A(\varphi) & A(p_1) & -A(p_1) \\ A(\varphi) & 0 & 0 & 0 & A(q_2) & -A(q_1) \\ 0 & A(\varphi) & 0 & 0 & A(q_1) & -A(q_2) \\ -A(p_2) & -A(p_1) & -A(q_2) & -A(q_1) & 0 & 0 \\ A(p_1) & A(p_1) & A(q_1) & A(q_2) & 0 & 0 \end{pmatrix} .$$

Hence, $\gamma K \alpha_1' = A(\varphi) + A(q_2) \neq 0$ and $\gamma K \alpha_2' = A(q_1) \neq 0$. This results in

$$\begin{aligned}\frac{1}{4}\mu_{s,\psi}(A(\varphi) + A(q_2))^2 \quad &\leq \quad \left[\Delta^2_{s,\psi}(A(p_1)) + \Delta^2_{s,\psi}(A(\theta))\right]\Delta^2_{s,\psi}(A(q_1)) \\ &\quad +\Delta^2_{s,\psi}(A(q_1))\left[\Delta_{s,\psi}(A(\theta), A(p_1)) + \overline{\Delta_{s,\psi}(A(\theta), A(p_1))}\right]\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}\frac{1}{4}\mu_{s,\psi}(A(q_1))^2 \quad &\leq \quad \left[\Delta^2_{s,\psi}(A(p_1)) + \Delta^2_{s,\psi}(A(\theta))\right]\Delta^2_{s,\psi}(A(q_2)) \\ &\quad +\Delta^2_{s,\psi}(A(q_2))\left[\Delta_{s,\psi}(A(\theta), A(p_1)) + \overline{\Delta_{s,\psi}(A(\theta), A(p_1))}\right] \ ,\end{aligned} \tag{4.21}$$

what exactly coincides with (4.19).

For arbitrary s the related partial differential equations become somehow complicated and hence, the computation of explicit solutions is really subtle. To this end, we limit the computation of minimizers of (4.20) and (4.21) to the special case $s = 0$.

Using the explicit expressions of $A(\theta)$, $A(p_1)$, and $A(q_l)$, namely

$$A(\theta)\hat{\psi} = i\nabla\hat{\psi}\cdot\begin{pmatrix}\omega_2\\-\omega_1\end{pmatrix}\ ,\quad A(p_1)\hat{\psi} = \frac{s\omega_1}{1+\|\omega\|^2}\hat{\psi} + i\frac{\partial}{\partial\omega_1}\hat{\psi}\ \text{ and }\ A(q_l)\hat{\psi} = \omega_l\hat{\psi}$$

and incorporating the structures of $A^\sigma(\theta)$ and $A^\sigma(q_l)$, we have to solve the following partial differential equations

$$i\nabla\hat{\psi}\cdot\begin{pmatrix}\omega_2\\-\omega_1\end{pmatrix} + i\frac{\partial}{\partial\omega_1}\hat{\psi} - \mu_1\hat{\psi} = it\left(\omega_l\hat{\psi} - \mu_2\hat{\psi}\right)\ ,\ l = 1,2. \tag{4.22}$$

Lemma 4.3 *For the indicated cases the function $\hat{\psi}$ satisfies the partial differential equation (4.22)*

$$\begin{aligned}
l=1,\ 1+\omega_2\geq 0 &: \quad \hat{\psi}(\omega) = \hat{\phi}\left(\frac{\|\omega\|^2}{2}+\omega_2\right)(i\omega_1+|1+\omega_2|)^{-\mu_1+it\mu_2}\,e^{-|1+\omega_2|t}\ ,\\
l=1,\ 1+\omega_2< 0 &: \quad \hat{\psi}(\omega) = \hat{\phi}\left(\frac{\|\omega\|^2}{2}+\omega_2\right)(i\omega_1+|1+\omega_2|)^{\mu_1-it\mu_2}\,e^{|1+\omega_2|t}\ ,\\
l=2,\ 1+\omega_2\geq 0 &: \quad \hat{\psi}(\omega) = \hat{\phi}\left(\frac{\|\omega\|^2}{2}+\omega_2\right)(i\omega_1+|1+\omega_2|)^{-\mu_1+(1+\mu_2)it}\,e^{\omega_1 t}\ ,\\
l=2,\ 1+\omega_2< 0 &: \quad \hat{\psi}(\omega) = \hat{\phi}\left(\frac{\|\omega\|^2}{2}+\omega_2\right)(i\omega_1+|1+\omega_2|)^{\mu_1-(1+\mu_2)it}\,e^{\omega_1 t}\ ,
\end{aligned}$$

where $\hat{\phi}\in C^1$ is some arbitrary function.

Proof For each case one can show that $\hat{\psi}$ satisfies (4.22). We start with $l = 1$ and consider the case $\omega_2 + 1 \geq 0$, i.e.

$$\hat{\psi}(\omega_1,\omega_2) = \hat{\phi}\left(\frac{\|\omega\|^2}{2}+\omega_2\right)(i\omega_1+|1+\omega_2|)^{-\mu_1+it\mu_2}\,e^{-|1+\omega_2|t}\ .$$

We obtain the following derivatives

$$\begin{aligned}
\frac{d}{d\omega_1}\hat{\psi}(\omega_1,\omega_2) &= \left(\omega_1\hat{\phi}' + (-i\mu_1 - t\mu_2)(i\omega_1+|1+\omega_2|)^{-1}\hat{\phi}\right)(i\omega_1+|1+\omega_2|)^{-\mu_1+it\mu_2}e^{-|1+\omega_2|t},\\
\frac{d}{d\omega_2}\hat{\psi}(\omega_1,\omega_2) &= \left((\omega_2+1)\hat{\phi}' + \mathrm{sgn}(\omega_2+1)(-\mu_1+it\mu_2)(i\omega_1+|1+\omega_2|)^{-1}\hat{\phi} - \mathrm{sgn}(\omega_2+1)t\hat{\phi}\right)\\
&\qquad \times(i\omega_1+|1+\omega_2|)^{-\mu_1+it\mu_2}e^{-|1+\omega_2|t}.
\end{aligned}$$

Inserting the derivatives in the left hand side of (4.22) (multiplied with $-i$) yields

$$\begin{aligned}
\text{l.h.s.} &= \Big[\Big(\mathrm{sgn}(\omega_2+1)|\omega_2+1|(-i\mu_1 - t\mu_2)\hat{\phi} + ix(-i\mu_1-t\mu_2)\mathrm{sgn}(\omega_2+1)\hat{\phi}\Big)\\
&\qquad \times(ix+|\omega_2+1|)^{-1} + t\omega_1\mathrm{sgn}(\omega_2+1)\hat{\phi}\Big]\,(i\omega_1+|1+\omega_2|)^{-\mu_1+it\mu_2}e^{-|1+\omega_2|t}\\
&= \mathrm{sgn}(\omega_2+1)(t\omega_1 - i\mu_1 - t\mu_2)\hat{\phi}(i\omega_1+|1+\omega_2|)^{-\mu_1+it\mu_2}e^{-|1+\omega_2|t}\\
&= \mathrm{sgn}(\omega_2+1)(t\omega_1 - i\mu_1 - t\mu_2)\hat{\psi}(\omega)\ ,
\end{aligned}$$

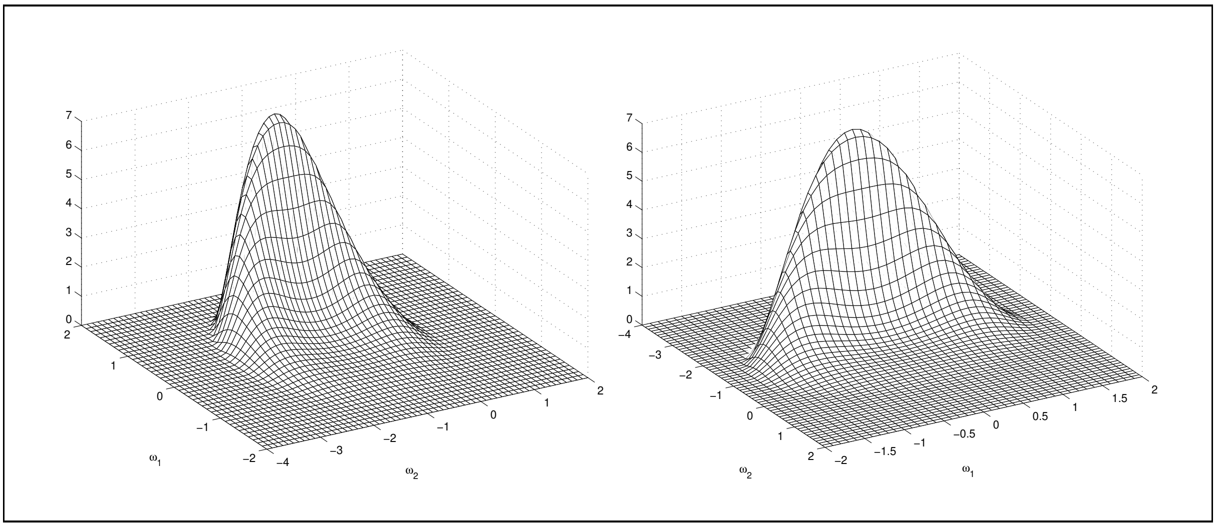

Figure 2: Wavelet functions based on $\mathbf{G}_{aWH}$. Left: $\hat{\psi}$ for $l = 1$, $(\mu_1, \mu_2) = (0, 0)$, $t = 2$; right: $\hat{\psi}$ for $l = 2$, $(\mu_1, \mu_2) = (0, -1)$, $t = 2$.

what coincides for $\omega_2 + 1 \geq 0$ with the r.h.s of (4.22). This result can be obtained for some arbitrary but differentiable function $\hat{\phi}$. For the case $\omega_2 + 1 < 0$ we proceed in the same way. Computing the derivatives of

$$\hat{\psi}(\omega) = \hat{\phi}\left(\frac{\|\omega\|^2}{2} + \omega_2\right)(i\omega_1 + |1 + \omega_2|)^{\mu_1 - it\mu_2}\, e^{|1+\omega_2|t}$$

and inserting them in the left hand side of (4.22) (multiplied with $-i$) yields

$$\begin{aligned} \text{l.h.s.} &= \operatorname{sgn}(\omega_2 + 1)(i\mu_1 + t\mu_2 - t\omega_1)\hat{\phi}(i\omega_1 + |1 + \omega_2|)^{\mu_1 - it\mu_2} e^{|1+\omega_2|t} \\ &= -\operatorname{sgn}(\omega_2 + 1)(t\omega_1 - i\mu_1 - t\mu_2)\hat{\psi}(\omega)\ , \end{aligned}$$

what again coincides with the r.h.s of (4.22), but now for $\omega_2 + 1 < 0$. The proof for $l = 2$ can be treated in the manner and is therefore omitted. ■

Finally, we have to find a suitable function $\hat{\phi}$ and adequate parameters μ_1, μ_2 and t such that $\hat{\psi}$ satisfies (4.11). The admissibility condition (4.11) means that $\hat{\psi}$ must vanish on the hyperplane $|1 - \langle v, \omega\rangle\,| = |1 - \omega_2| = 0$. For $l = 1, 2$ almost every choice of μ_1 and μ_2 cause some problems. For our purposes a somewhat trivial setting fits with (4.11), namely $(\mu_1, \mu_2)_{l=1} = (0, 0)$, $(\mu_1, \mu_2)_{l=2} = (1, -1)$, and $\hat{\phi}(\tau) = (\cos(3\tau) + 1) \cdot \chi_{[-\pi,\pi]}(3\tau)$. The resulting wavelets are visualized in Figure 2.

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Appendix

Tensor Product Hilbert Spaces

We repeat some basic facts about tensor product Hilbert spaces. Up to equivalence there are three different possibilities to construct a tensor product of two Hilbert spaces, see [DF93]. We use a method presented in [Hoc98]. Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two given Hilbert spaces and $\mathcal{H}_1'$ and $\mathcal{H}_2'$ are its duals. Let $\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$ be the space of bilinear mappings. Then, the tensor product $x \otimes y$ is defined as the element in $\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$ with

$$(x \otimes y)(x', y') := x'(x)y'(y) \ , \quad (x', y') \in \mathcal{H}_1' \times \mathcal{H}_2' \ ,$$

where $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$. Consequently, $\mathcal{H}_1 \otimes \mathcal{H}_2$ is generated by tensor products $x \otimes y$, where $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$. Hence, this space is defined as a subspace of $\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$. Furthermore, we need to establish a "inner product" in this space. Let $f, g \in \mathcal{H}_1 \otimes \mathcal{H}_2$, where $f = \sum_{j=1}^{m_j} c_j u_j \otimes v_j$ and $g = \sum_{k=1}^{m_k} d_k w_k \otimes z_k$ and $u_j, w_k \in \mathcal{H}_1$, $v_j, z_k \in \mathcal{H}_2$. By

$$l(f, g) := \sum_{j=1}^{m_j} \sum_{k=1}^{m_k} c_j d_k \left\langle u_j, w_k \right\rangle_{\mathcal{H}_1} \left\langle v_j, z_k \right\rangle_{\mathcal{H}_2}$$

we define a sesqui-linear functional. Equipped with this sesqui-linear functional, $\mathcal{H}_1 \otimes \mathcal{H}_2$ becomes a pre-Hilbert space. The tensor product

$$\mathcal{H}_1 \otimes_l \mathcal{H}_2$$

is then defined as the closure of $\mathcal{H}_1 \otimes \mathcal{H}_2$ with respect to $l(\cdot, \cdot)$.

Proof of Lemma 4.1

Assume $s_1 = s \geq 0$ and $s_2 = t \geq 0$. Then its obvious that $\pi_{s,t}$ is a homomorphism:

$$\begin{aligned}
\pi_{s,t}((g, h) \circ (g', h')) &= \pi_{s,t}(g \circ g', h \circ h') = \pi_{1,s}(g \circ g') \otimes \pi_{2,t}(h \circ h') \\
&= \pi_{1,s}(g)\pi_{1,s}(g') \otimes \pi_{2,t}(h)\pi_{2,t}(h') \\
&= (\pi_{1,s}(g) \otimes \pi_{2,t}(h))(\pi_{1,s}(g') \otimes \pi_{2,t}(h')) \\
&= \pi_{s,t}(g, h)\pi_{s,t}(g', h')
\end{aligned}$$

We know that $\pi_{1,s}$, the representation of $\mathbf{G}_{WH}$, and that $\pi_{2,t}$, the representation of $\mathbf{G}_{aff}$, are continuous. Hence, we conclude that $\pi_{s,t}$ continuous. To show that $\pi_{s,t}$ is square integrable one has to show that $\pi_{s,t}$ is irreducible and there exists a non-trivial vector. Assume $\pi_{s,t}$ is reducible. This means there exists a closed, non-trivial subspace $V \subseteq H^s(\mathbb{R}^m) \otimes_l H^t(\mathbb{R}^n)$ with $\pi_{s,t}(g)V \subseteq V \ \forall g \in \mathbf{G}_{aWH}^d$. Under this assumption there exist non-trivial functions $h \in V$ and $f \in V^\perp$.
Let $f, h \in H^s(\mathbb{R}^m) \otimes_l H^t(\mathbb{R}^n)$ and

$$(\Lambda_{s,t} f)^\wedge (k_1, k_2) = (1 + \|k_1\|^2)^{s/2}(1 + \|k_2\|^2)^{t/2} \hat{f}(k_1, k_2) \ .$$

Then, we have by $\Lambda_{s,t} f =: F$ and $\Lambda_{s,t} h =: H$

$$\left\langle \pi_{s,t}(g)f, h \right\rangle_{H^s(\mathbb{R}^m) \otimes_l H^t} = \left\langle \pi(g)F, H \right\rangle_{L_2(\mathbb{R}^{m+n})} = \left\langle \hat{\pi}(g)\hat{F}, \hat{H} \right\rangle_{L_2(\mathbb{R}^{m+n})} \ .$$

Because of

$$\hat{\pi}(g)\hat{F}(\omega, \eta) = a^{n/2} e^{i\varphi} e^{-i(\omega,\eta)(q,b)'} \hat{F}(\omega + p, aR'\eta)$$

it follows that

$$\left\langle \hat{\pi}(g)\hat{F}, \hat{H} \right\rangle_{L_2(\mathbb{R}^{m+n})} = a^{n/2} e^{i\varphi} (2\pi)^{\frac{m+n}{2}} \mathcal{F}\left(K(\cdot,\cdot,p,a,R)\right)(q,b) \ , \tag{4.23}$$

where $K(\omega,\eta,p,a,R) = \hat{F}(\omega+p, aR'\eta)\bar{\hat{H}}(\omega,\eta)$. By $\Omega = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}_+ \times SO(n)$ and some substitutions we obtain

$$\int\limits_{\mathbf{G}^d_{aWH}} |\left\langle \pi(p,q,\varphi,b,a,R)F, H \right\rangle_{L_2}|^2 d\mu d\nu = (2\pi)^{m+n+1} \int\limits_{\mathbb{R}^{m+n}} |\hat{H}(\omega,\eta)|^2 I(\eta) d\omega d\eta \ , \tag{4.24}$$

where

$$I(\eta) = \int\limits_{\mathbb{R}^m \times \mathbb{R}_+ \times SO(n)} |\hat{F}(\xi, aR'\eta)|^2 a^{-1} d\xi da dm(R) \ .$$

Using Euler-angle representation of $R \in SO(n)$ and some standard substitutions $I(\eta)$ can be expressed by

$$I(\eta) = vol(SO(n-1)) \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^n} \frac{|\hat{F}(\xi,\zeta)|^2}{\|\zeta\|^n} d\zeta d\xi \ .$$

Hence, (4.24) is equal to

$$(2\pi)^{m+n+1} \cdot vol(SO(n-1)) \cdot \|h\|^2_{s,t} \cdot \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^n} (1+\|\xi\|^2)^s (1+\|\zeta\|^2)^t \frac{|\hat{f}(\xi,\zeta)|^2}{\|\zeta\|^n} d\zeta d\xi \ .$$

Assume that $\pi_{s,t}$ is reducible. Then, there exist vectors

$$0 \neq h \in V \subset H^s(\mathbb{R}^m) \otimes_l H^t(\mathbb{R}^n) \ \text{ and } \ 0 \neq f \in V^\perp$$

such that

$$0 = \left\langle \pi(p,q,\varphi,b,a,R)F, H \right\rangle_{L_2} \quad .$$

But this implies that

$$0 = \|h\|^2_{s,t} \cdot \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^n} (1+\|\xi\|^2)^s (1+\|\zeta\|^2)^t \frac{|\hat{f}(\xi,\zeta)|^2}{\|\zeta\|^n} d\zeta d\xi$$

is true. But this is contradictory to $0 \neq f, h$. If $\hat{f}$ vanishes in a neighborhood of the origin and setting $f = h \neq 0$ it follows that

$$0 < \|f\|^2_{s,t} \cdot \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^n} (1+\|\xi\|^2)^s (1+\|\zeta\|^2)^t \frac{|\hat{f}(\xi,\zeta)|^2}{\|\zeta\|^n} d\zeta d\xi < \infty.$$

Such a function f evidently exists. ■