SAMPLING THEORY IN SIGNAL AND IMAGE PROCESSING Vol. 3, No. 2, May 2004, pp. 175-196

ISSN: 1530-6429

Regularization of Sobolev Embedding Operators and Applications Part I: Fourier and Wavelet based Methods

Ronny Ramlau and Gerd Teschke
Universität Bremen, Zentrum für Technomathematik
Postfach 33 04 40
28334 Bremen
Germany
ramlau@math.uni-bremen.de, teschke@math.uni-bremen.de

Abstract

In this paper we present a smoothing technique for noisy data by the inversion of Sobolev embedding operators. We assume that the true data x belong to a Sobolev space H^s with known smoothness index s, and that the measurements x^δ of x belong to L_2 (or to H^t with $t < s$) only. Our main goal is to compute a H^s approximation of the true data. We will use Tikhonov regularization or the stable inversion of the embedding operator, and show how wavelet and Fourier methods can be effectively used to compute a pointwise approximation to the inverse embedding operator. In the forthcoming paper *Regularization of Sobolev Embedding Operators and Applications Part II: Data Driven Regularization and Applications* we will propose a data driven rule for the regularization parameter and we prove order optimal convergence of the resulting regularization (smoothing) scheme.

Key words and phrases : Data smoothing, Sobolev embedding operator, Tikhonov regularization

2000 AMS Mathematics Subject Classification — 65J10, 65J20, 65J22, 94A12, 94A20

1 Introduction

One important question in data processing is the reconstruction of a true signal from an observation. In almost all applications the observation is a noisy and blurred version of the true signal or image. In order to extract the underlying information from the data, it is often necessary to use denoising/restoration algorithms beforehand. In recent years, several algorithms for data preprocessing have been proposed. There exists a abundant literature dealing with stochastic, wavelet and Fourier based preprocessing methods, etc., see, e.g., [7, 8, 9, 14, 16, 21].

The main result of this paper is the construction of a smoothing scheme based on the inversion of Sobolev embedding operators. We start with the observation that it is often theoretically known that the data satisfy some smoothness conditions. In our framework, we will assume that the data x belong to a Sobolev space H^s with smoothness index $s > 0$. The measured data x^δ will usually only belong to L_2 (or to a Sobolev space of less smoothness), i.e. the Sobolev norm of $x - x^\delta$ can be arbitrary large. If $i : H^s \to L_2$ denotes the embedding operator, then a typical measurement process can be described as follows:

$$x \in H^s \mapsto ix + noise = x^\delta \in L_2 \ .$$

Thus if we want to construct an approximation $\tilde{x}$ to x in H^s, we have to solve the equation

$$ix = x \tag{1.1}$$

from noisy data x^δ. As it is well known that i is an ill-posed operator, we have to apply regularization methods. We will particularly focus on Tikhonov-regularization. For computing the approximation $\tilde{x}$, we propose wavelet and Fourier based techniques. Moreover, for regularization methods, the quality of the reconstruction depends mainly on the chosen regularization parameter. In the classical theory, the choice of this parameter is related to the error level δ with $\|x - x^\delta\|_{L_2} \leq \delta$. However, in practical applications, δ is only roughly or not known. To overcome this deficiency we propose in [22] a data driven parameter rule without knowledge of the data error level.

We remark that the regularization of the embedding operator was used before for the inversion of some types of linear integral equations. Bruckner [1] used a truncated singular value decomposition for the regularization of the embedding operator in connection with the inversion of an integral operator with logarithmic kernel. In [2] interpolating function systems were used for the regularization of the compact embedding operator i. In contrast, we will particularly focus on the embedding operator on unbounded domains (where i is not compact).

As mentioned before, we will consider Tikhonov regularization for the regularization of the embedding operator only; methods like Landweber or conjugate gradient will give similar results. Other methods for signal and image denoising include diffusion filtering by partial differential equations. For a good overview, see [23]. In this context, we might refer to the approach by Groetsch and Scherzer [12], where the authors investigated a nonstationary iterated Tikhonov-Morozov method for the stable evaluation of unbounded operators D with noisy data $x^\delta \notin D(L)$. Setting $D_s = (1 + \Delta)^{s/2}$, it is well known that $x \in H^s$ iff $D_s x \in L_2$. Now, for noisy data $x^\delta \in L_2$, the application of D_s to x^δ will be in general unstable, but the nonstationary iterated Tikhonov-Morozov method gives an approximation $\tilde{x}$ to x in the H^s sense as well as an approximation $D_s\tilde{x}$ to $D_s x$. However, as the nonstationary method requires the evaluation of a partial differential equation within every iteration step it seems to us that our method is numerically advantageous in case of the embedding operator.

This paper is organized as follows: In Section 2 we summarize known results on the theory of regularization methods and on wavelets. Section 3 is mainly concerned with the computation of the minimizer of the Tikhonov-functional. We propose two methods based on a wavelet based discretization of the embedding operator and on a Fourier representation of the adjoint embedding operator. In order to obtain convergence rates, a smoothness condition $x = (i^*i)^{\nu/2}\omega$ is required; we will give a simple criterion for this condition. In the second part [22] we switch to a statistical setting and introduce our data driven parameter rule and give a proof for the optimality of the method. We illustrate our results with some applications arising in the context of medical imaging and meteorological signal processing.

2 Preliminaries

The analysis of the following sections is mainly based on the theory of linear ill-posed problems and on some properties of wavelet systems. Thus we need to summarize some well known facts beforehand.

2.1 Regularization methods for linear inverse problems

The theory of linear ill-posed problems has been extensively studied; for a good overview on the topic, we refer to the monographs [15, 10]. Let $A : X \to Y$ be a linear operator between Hilbert spaces, and assume that we are looking for a solution of the equation

$$Ax = y \ . \tag{2.1}$$

If the range of A, $R(A)$, is not closed, then A^{-1} is an unbounded operator. If the right hand side in (2.1) stems from measurements, then we can only hope for noisy data y^δ with

$$\|y^\delta - y\| \le \delta \ . \tag{2.2}$$

In this case, $x^\delta := A^{-1}y^\delta$ might be an arbitrary bad approximation to a solution of (2.1). To obtain a stable solution, one has to use regularization methods, where the inverse operator A^{-1} is (pointwise) approximated by a family of operators T_α. The regularization parameter $\alpha(\delta, y^\delta)$ has to be chosen such that for y^δ with (2.2) and $y \in R(A)$

$$\lim_{\delta\to 0} T_{\alpha(\delta,y^\delta)} y^\delta = A^{-1}y \tag{2.3}$$

holds. With a given parameter choice rule, the convergence of $x_\alpha^\delta = T_\alpha y^\delta$ to a solution x_* of (2.1) might be arbitrary slow. To get error estimates, one has to use additional information to the solution x_*. A common assumption is

$$x_* = (A^*A)^{\nu/2}\omega \ , \tag{2.4}$$

$\nu > 0$. A regularization method will then be called *order optimal*, if the error estimate

$$\|x_* - x_\alpha^\delta\| = O(\delta^{\nu/(\nu+1)}) \tag{2.5}$$

In this paper, we will focus on Tikhonov-regularization only. For a given α, the regularized solution is computed as the minimizing function of the Tikhonov-functional

$$J_\alpha(x) = \|y^\delta - Ax\|^2 + \alpha\|x\|^2 \ . \tag{2.6}$$

It is easy to see that the minimizing function x_α^δ can be obtained as the solution of the linear system

$$(A^*A + \alpha I)x = A^*y^\delta \ , \tag{2.7}$$

and the operator T_α is therefore given by $T_\alpha = (A^*A+\alpha I)^{-1}A^*$. Parameter choice rules for Tikhonov–regularization are well studied. We will use Morozov's discrepancy principle, where the regularization parameter α is chosen such that $\|y^\delta - Ax^\delta_\alpha\| = c\delta$, $c \geq 1$ holds. For a numerical realization, the following modification is often used [15, 11, 18]: Choose $c_1,\ \alpha_0 > 0$, $0 < q < 1$, $\alpha_j = q^j\alpha_0$, and compute $x^\delta_{\alpha_j}$ until

$$\delta \leq \|y^\delta - Ax^\delta_{\alpha_j}\| \leq c_1\delta \tag{2.8}$$

holds. It is well known that Tikhonov–regularization combined with Morozov's discrepancy principle, is an order optimal method. We wish to remark that an a priori knowledge of the error δ is necessary for the discrepancy principle. The quality of the inversion will strongly depend on an accurate knowledge of δ. Therefore, we will focus in [22] on the development of a regularization parameter choice rule that estimates the noise level out of the data.

In some sense, regularization methods can be interpreted as the application of the generalized inverse $A^\dagger$ of A to the pre-smoothed data. This fact is easily seen for compact operators. If A has a singular system $\{u_n,\ v_n,\ \sigma_n\}$ with $Af = \sum \sigma_n\langle f, u_n\rangle v_n$, then linear regularization methods can be defined by using a filter function $F_\alpha(\sigma)$, and $T_\alpha y^\delta$ is given by

$$T_\alpha y^\delta = \sum_n F_\alpha(\sigma_n)\sigma_n^{-1}\langle y^\delta, v_n\rangle u_n\ .$$

Although the data y^δ belongs usually not to $R(A)$, we can "smooth" the data by applying the operator S_α, defined by $S_\alpha y^\delta = \sum F_\alpha\langle y^\delta, v_n\rangle v_n$. If the filter F_α is regularizing (cf. [15]), then $S_\alpha y^\delta \in R(A)$, and T_α can be rewritten as $T_\alpha = A^\dagger S_\alpha$. We might call this concept *regularizing by data smoothing*, and the goal of our paper will then be the investigation of the converse method, *data smoothing by regularization*.

2.2 Some basic facts on wavelets

One of the methods to solve the inverse Sobolev embedding problem is a wavelet based discretization scheme. To this end, it becomes necessary to recall some basic facts on wavelets. For a comprehensive review on wavelets we would refer the reader to the very rich literature, e.g. [6, 7, 17]. Here we summarize only the results needed later on.

Suppose H is a Hilbert space and the sequence $\{V_j\}$ forms a multi-resolution analysis for H, where we assume that V_j are spanned by $\Phi_j = \{\phi_{j,k} : k \in I_j\}$ which are uniformly stable. Successively updating a current approximation in V_j to a better one in V_{j+1} can be facilitated if stable bases $\Psi_j = \{\psi_{j,k} : k \in J_j\}$ for some complement W_j of V_j in V_{j+1} are available. Then, any $f_j \in V_j$ has then an alternative multi-scale representation

$$f_j = \sum_{k\in I_0} c_{0,k}\phi_{0,k} + \sum_{j'=0}^{j}\sum_{k\in J_{j'}} d_{j',k}\psi_{j',k}.$$

The essential constraint on the choice of W_j is that $\Psi = \bigcup_j \Psi_j$ forms a Riesz-basis of H (as well as the related bi-orthogonal system $\tilde{\Psi}$), cf. [3, 7]. With such a pair of bi-orthogonal bases Ψ and $\tilde{\Psi}$ one can associate canonical truncation projectors

$$Q_j f := \sum_{j'=0}^{j}\sum_{k\in J_{j'}} (f, \tilde{\psi}_{j',k})\psi_{j',k}, \quad \tilde{Q}_j f := \sum_{j'=0}^{j}\sum_{k\in J_{j'}} (f, \psi_{j',k})\tilde{\psi}_{j',k}$$

which are obviously adjoints of each other. Of course, when Ψ is a Riesz-basis then the Q_j and hence their adjoints $\tilde{Q}_j$ are uniformly bounded in H. Denoting by $\tilde{V}_j$ the range of $\tilde{Q}_j$ we have therefore two sequences of nested closed subspaces V_j and $\tilde{V}_j$, respectively, whose union is easily seen to be dense in H, see [3].

What turns out to matter is that both $\{V_j\}$ and $\{\tilde{V}_j\}$ should have some approximation and regularity properties which can be stated in terms of the following pair of estimates. There exists some $\gamma > 0$ such that the inverse estimate

$$\|f_j\|_{H^s} \lesssim 2^{js}\|f_j\|_{L_2}, \quad f_j \in V_j, \tag{2.9}$$

holds for $s < \gamma$. Moreover, there exists some $m \geq \gamma$ such that the direct estimate

$$\inf_{f_j\in V_j} \|f - f_j\|_{L_2} \lesssim 2^{-js}\|f\|_{H^s}, \quad f \in H^s, \tag{2.10}$$

holds for $s \leq m$. The next Theorem, proved in [4], provides us with H^s stability of L_2 bi-orthogonal and smooth wavelet systems.

Theorem 2.1 *Suppose that* $\Psi = \{\psi_\lambda : \lambda \in J\}$ *and* $\tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in J\}$ *are bi-orthogonal collections in* L_2 *and that the associated sequence of projectors* $\{Q_j\}_{j=0}^{\infty}$ *is uniformly bounded. If both* $\{Q_j\}$ *and* $\{\tilde{Q}_j\}$ *satisfy (2.9) and (2.10) relative to some* $\gamma, \gamma' > 0$, $\gamma \le m$, $\gamma' \le m'$, *then*

$$\begin{aligned} \|f\|_{H^s(\Omega)} &\sim \left(\sum_{\lambda \in J} 2^{2|\lambda| s} |\langle f, \tilde{\psi}_\lambda \rangle|^2\right)^{\frac{1}{2}}, \quad s \in (-\gamma', \gamma), \qquad (2.11) \\ &\sim \left(\sum_{\lambda \in J} 2^{2|\lambda| s} |\langle f, \psi_\lambda \rangle|^2\right)^{\frac{1}{2}}, \quad s \in (-\gamma, \gamma'), \quad f \in H^s. \end{aligned}$$

Moreover, the projectors Q_j, $\tilde{Q}_j$ *are uniformly bounded in* H^s, $s \in (-\gamma', \gamma)$ *and* $s \in (-\gamma, \gamma')$, *respectively.*

Beside the H^s stability of wavelet systems the self similarity is an important property for our purpose. The next theorem shows that computing the inner products of nodal functions ϕ (or its derivatives) is equivalent with solving an eigenvalue problem. For comprehensive details we refer the reader to [6].

Theorem 2.2 *Let* ϕ^l, $l = 0, 1$, *be compactly supported refinable functions which are* ℓ^∞*–stable and belong to* $C^d(\mathbb{R})$ *and let* $\{h_\nu^l\}$ *denote the corresponding filter masks. Then for every* $\mu \in \mathbb{N}$, $\mu \le d$, *there exists a unique sequence* $\mathcal{V} = \{\mathcal{V}_\beta,\ \beta \in \mathbb{Z}\}$ *of finite support satisfying*

$$2^{-\mu} \mathcal{V}_\alpha = \sum_{\beta \in \mathbb{Z}} c_{2\eta - \beta} \mathcal{V}_\beta, \quad \eta \in \mathbb{Z}\ ,$$

where c *is given by*

$$c_\mu = 2^{-1} \sum_{\nu \in \mathbb{Z}} h_\nu^0 h_{\nu-\mu}^1\ .$$

Moreover, $\mathcal{V}$ *is given by*

$$\mathcal{V}_\alpha = (-1)^\mu \langle \phi^0, (\phi^1)^{(\mu)}(\cdot - \eta) \rangle_{L_2}\ .$$

3 Regularization of the embedding operators

In this section we will use a regularization method, namely Tikhonov regularization for the denoising of measured data. We assume that the true data x belong to H^s. Due to measurement noise, the measured data x^δ will usually only belong to L_2, and the task is to compute an approximation (in the H^s-sense) x_α^δ to the true data. To this end, we have to invert the embedding $i : H^s \to L_2$, and as i is ill posed, we have to use regularization methods. Here, we will use Tikhonov–regularization for the inversion of the embedding operator. To find a minimizer of the Tikhonov–functional

$$J_\alpha(x) = \|x^\delta - i(x)\|^2 + \alpha \|x\|_s^2 \ , \tag{3.1}$$

we have to solve the equation

$$(i^* i + \alpha I)x = i^* x^\delta \ . \tag{3.2}$$

In particular, we have to derive a representation of the adjoint i^* embedding operator. To obtain convergence rates, a range condition $x_* = (i^* i)^{\nu/2}\omega$ is required; we are going to give a simple characterization of this range condition.

A common definition of the Sobolev spaces uses the Fourier transform, and thus it seems reasonable to use Fourier-techniques for the solution of (3.2). It turns out that the computational effort of this approach is relatively small (in principle, the Fourier transform has to be applied twice). In this approach, the solution of (3.2) is computed in the frequency domain.

On the other hand, if the Sobolev–index s is small, then the Fourier transform of the solution x_* decreases slowly, meaning that we will need the Fourier-transform of the data for many frequencies in order to reconstruct the solution. If this information is not available, it might be more suitable to consider (3.2) in the time domain. For a numerical realization, we have to consider this equation in a finite dimensional setting. By choosing an n dimensional (wavelet-) ansatz space V_n, we compute an approximation $x_{\alpha,n}^\delta$ to the solution x_α^δ of (3.2) by solving a matrix-vector equation, and estimate the distance between $x_{\alpha,n}^\delta$ and x_α^δ.

3.1 Regularization by continuous Fourier transform

In the following, we will consider the embedding operator

$$i : H^s(\mathbb{R}) \to L_2(\mathbb{R}) \ , \ s > 0 \ . \tag{3.3}$$

It is well known that the operator i is continuous, but not compact. If we want to use Tikhonov–regularization, then we have to know the adjoint operator i^* of i.

Lemma 3.1 *Let i be defined as in (3.3). Then the adjoint operator*

$$i^* : L_2(\mathbb{R}) \to H^s(\mathbb{R}) \tag{3.4}$$

is defined in the Fourier space by

$$\widehat{(i^*z)}(\omega) = \frac{\hat{z}(\omega)}{(1+|\omega|^2)^s} \ . \tag{3.5}$$

If we have for $x \in H^s(\mathbb{R})$ only noisy data x^δ with $\|x^\delta - x\| \le \delta$ available, then the minimizer x_α^δ of the Tikhonov–functional

$$J_\alpha(x) = \|ix - x^\delta\|^2 + \alpha\|x\|_s^2$$

is given by

$$\hat{x}_\alpha^\delta(\omega) = \frac{\widehat{x^\delta}(\omega)}{1 + \alpha\left((1+|\omega|^2)^s\right)} \ . \tag{3.6}$$

Proof Let $x \in H^s(\mathbb{R})$ and $z \in L_2(\mathbb{R})$ be given. Then we have

$$\int_{\mathbb{R}} \hat{x}(\omega)\hat{z}(\omega)\,d\omega \;=\; \langle ix, z\rangle_{L_2} = \langle x, i^*z\rangle_{H^s} = \int_{\mathbb{R}} (1+|\omega|^2)^s \hat{x}(\omega)\widehat{(i^*z)}(\omega)\,d\omega \ .$$

Thus we obtain

$$\int_{\mathbb{R}} \hat{x}(\omega)\left((1+|\omega|^2)^s\widehat{(i^*z)}(\omega) - \hat{z}(\omega)\right)\,d\omega = 0 \tag{3.7}$$

for all $x \in H^s(\mathbb{R})$, i.e.

$$\widehat{(i^*z)}(\omega) = \frac{\hat{z}(\omega)}{(1+|\omega|^2)^s} \ .$$

The minimizer of the Tikhonov–functional is given by the solution of the equation

$$(i^*i + \alpha I)x = i^*x^\delta \ .$$

Using $i^*i = i^*$ we get by (3.5)

$$\frac{\hat{x}(\omega)}{(1+|\omega|^2)^s} + \alpha\hat{x}(\omega) = \frac{\widehat{x^\delta}(\omega)}{(1+|\omega|^2)^s} ,$$

which is equivalent to (3.6). ∎

Setting

$$\sigma(\omega) = \frac{1}{(1+|\omega|^2)^s} , \tag{3.8}$$

we obtain by (3.5)

$$i^*i = \mathcal{F}^{-1}(\sigma\mathcal{F}) , \tag{3.9}$$

i.e. i^*i is a pseudo–differential operator with symbol σ. As a function of a pseudo–differential operator is usually defined via its symbol,

$$f(i^*i) := \mathcal{F}^{-1}(f(\sigma)\mathcal{F}) ,$$

the operator $(i^*i)^{\nu/2}$ is defined by

$$\mathcal{F}\left((i^*i)^{\nu/2}z\right)(\omega) = \frac{\hat{z}(\omega)}{(1+|\omega|^2)^{s\nu/2}} \tag{3.10}$$

Now we can characterize the range condition (2.4) by a smoothness condition in terms of a Sobolev scale.

Lemma 3.2 *Let i denote the embedding operator (3.3). Then a smoothness condition $x = (i^*i)^{\nu/2}z$ holds if and only if $x \in H^{s+s\nu}$.*

Proof It is

$$\hat{x}(\omega) = \mathcal{F}\left((i^*i)^{\nu/2}z\right)(\omega) = \frac{\hat{z}(\omega)}{(1+|\omega|^2)^{s\nu/2}} ,$$

or equivalently,

$$\hat{z}(\omega) = \hat{x}(\omega)(1+|\omega|^2)^{s\nu/2} . \tag{3.11}$$

As $z \in H^s(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} (1+|\omega|^2)^s \hat{z}^2(\omega)\, d\omega < \infty ,$$

we have with (3.11)

$$\int_{\mathbb{R}} \hat{x}(1+|\omega|^2)^{s\nu}(1+|\omega|^2)^s \hat{x}^2(\omega)\, d\omega < \infty \ , \tag{3.12}$$

which is equivalent to $x \in H^{s+s\nu}(\mathbb{R})$. ■

Altogether, we can summarize our results in the following Theorem on convergence rates for Tikhonov–regularization:

Theorem 3.1 *Let the embedding $i : H^s(\mathbb{R}) \to L_2(\mathbb{R})$ be given by $i(x) = x$, x^δ be a noisy version of $x \in H^s(\mathbb{R})$ with $\|x - x^\delta\|_0 \leq \delta$, and x_α^δ be the minimizing element of the Tikhonov–functional (3.1), where the regularization parameter is determined via Morozov's discrepancy principle. If $x \in H^{s+s\nu}$, then the error estimate*

$$\|x_\alpha^\delta - x\| = O(\delta^{\nu/(\nu+1)}) \tag{3.13}$$

holds.

3.2 Sobolev embedding operators and discretizations

Next, we want to compute the minimizer of the Tikhonov functional in the time domain. This is in particular useful if the solution x_* belongs only to a Sobolev space with small smoothness index s. As stated above, the Fourier transform of x_* decreases slowly. In order to obtain a good reconstruction for the solution, we would have to evaluate (3.6) for many frequencies, which is difficult if we have not enough samples of the data.

Again, the minimization of the Tikhonov–functional

$$J_\alpha(x) = \|ix - x^\delta\|^2 + \alpha\|x\|^2$$

requires the evaluation of the adjoint of the embedding operator, i^*; for the numerical realization, this will be done by using a finite ansatz space. For a bounded domain $\Omega \subset \mathbb{R}$ let

$$B_n := \{e_1, ..., e_n\}_{n\in\mathbb{N}}$$

be an orthonormal system in $L_2(\Omega)$, $B_n^\perp$ the orthogonal complement of B_n in $L_2(\Omega)$ and $\{e_l^\perp\}_{n\in\mathbb{N}}$ an orthonormal system for $B_n^\perp$. Moreover, we may assume that e_l and $e_l^\perp$ belong to $H^s(\Omega)$ for all l. By $P_n : L_2(\Omega) \to B_n$ we denote the orthogonal projection from $L_2(\Omega)$ in B_n, and define a discretization i_n of i by

$$i_n := P_n i \ . \tag{3.14}$$

As all e_l belong to $H^s(\Omega)$, we have $P_n f \in H^s(\Omega)$ for all $f \in L_2(\Omega)$. The adjoint of i_n is given by

$$i_n^* = i^* P_n^* = i^* P_n \ .$$

Now, for given $g \in L_2(\Omega)$ let us determine $f = i_n^* g$. The function f can be decomposed into

$$f = \sum_{l=1}^n f_l e_l + \sum_l f_l^\perp e_l^\perp \ .$$

We observe with

$$P_n g = \sum_{l=1}^n g_l e_l$$

that

$$\begin{aligned} \langle f, e_k^\perp \rangle_{H^s} &= \langle i_n^* g, e_l^\perp \rangle_{H^s} = \langle P_n g, e_l^\perp \rangle_{L_2} \\ &= \sum_{l=1}^n g_l \langle e_l, e_l^\perp \rangle_{L_2} = 0 \ . \end{aligned} \tag{3.15}$$

Every $\tilde{f} \in H^s(\Omega) \cap B_n^\perp$ has a decomposition $\tilde{f} = \sum_l \tilde{f}_l e_l^\perp$, and by (3.15) we find

$$\langle f, \tilde{f} \rangle_{H^s} = \sum_l \tilde{f}_l \langle f, e_l^\perp \rangle_{H^s} = 0$$

for all $\tilde{f} \in H^s(\Omega) \cap B_n^\perp$. Thus $f = i_n^* g \in B_n$ and has therefore a decomposition $f = \sum_{l=1}^n f_l e_l$, and

$$\begin{aligned} \langle f, e_k \rangle_{H^s} &= \sum_{l=1}^n f_l \langle e_k, e_l \rangle_{H^s} = \langle i^* g, e_k \rangle_{H^s} \\ &= \sum_{l=1}^n g_l \langle e_l, e_k \rangle_{L_2} \end{aligned} \tag{3.16}$$

Setting

$$\bar{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} , \quad \bar{g} = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} , \quad E_{s,n} = (\langle e_l, e_k \rangle_{H^s})_{l,k=1,\ldots,n} \ , \text{ and}$$

$$E_n = (\langle e_l, e_k \rangle_{L_2})_{l,k=1,\ldots,n}$$

we have found

$$\bar{f} = E_{s,n}^{-1} E_n \bar{g} \ . \tag{3.17}$$

In case of an orthogonal system $\{e_l\}_{l=1,..,n}$ we observe $E = Id$. The above results will be summarized in the following

Proposition 3.1 *Let i_n be the discretization of the embedding operator i, defined in (3.14). The coefficients of the minimizing function x_α^δ of the Tikhonov-functional*

$$J_\alpha(x) = \|i_n x - x^\delta\|^2 + \alpha \|x\|^2$$

(with respect to the basis $\{e_l\}$ of B_n) are computed by

$$\bar{x}_{\alpha,n}^\delta = (E_{s,n}^{-1} E_n + \alpha I_n)^{-1} E_{s,n}^{-1} E_n x^\delta \ . \tag{3.18}$$

Here, I_n denotes the identity in B_n.

Proof Follows by (2.7) and (3.17). ■

The function $x_{\alpha,n}^\delta$ is an approximation of $x_\alpha^\delta = (i^* i + \alpha I)^{-1} i^* \, x^\delta$. In a next step, we want to estimate the distance between $x_{\alpha,n}^\delta$ and x_α^δ. To this end, we assume that P_n satisfies for $x \in H^s(\Omega)$ the following direct estimate

$$\|(P_n - I)x\|_{L_2(\Omega)} \leq 2^{-sm(n)} \|x\|_{H^s(\Omega)} \ , \tag{3.19}$$

where $m(n)$ is some constant depending on the index n. Moreover, we will use the following well-known result, for details and a proof, we refer the reader to [13].

Proposition 3.2 *Let x_n and x be solutions of the operator equations*

$$Ax = y \qquad \text{and} \qquad A_n x_n = y_n$$

with $A, A_n : X \to Y$ Assume that A^{-1} is bounded and that $\|A_n - A\| \to 0$ for $n \to \infty$ holds. Then we obtain for all $n \in \mathbb{N}$ with $\|A^{-1}(A_n - A)\| < 1$ the error estimate

$$\|x_n - x\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}(A_n - A)\|} \left\{ \|(A_n - A)x\| + \|y_n - y\| \right\} \ . \tag{3.20}$$

Combining assumption (3.19) and Proposition 3.2 we obtain an upper bound for the distance $x^\delta_{\alpha,n} - x^\delta_\alpha$.

Proposition 3.3 *Let the regularization parameter α be given, and choose $n \in \mathbb{N}$ such that*

$$\frac{1}{\alpha} 2^{1-m(n)s} < 1 \tag{3.21}$$

holds (for some $s \geq 0$). Furthermore, assume that the orthogonal projector P_n satisfies the direct estimate (3.19). Then the distance between x^δ_α and $x^\delta_{\alpha,n}$ can be estimated by

$$\|x^\delta_{\alpha,n} - x^\delta_\alpha\| \leq \frac{2+\sqrt{\alpha}}{\sqrt{\alpha}} \frac{2^{-sm(n)}}{\alpha - 2^{1-sm(n)}} \|x^\delta\| \ . \tag{3.22}$$

Proof According to $i_n^* x \in B_n$ we observe that the equations $(i_n^* i_n + \alpha I_n)x = i_n^* x^\delta$ and $(i_n^* i_n + \alpha I)x = i_n^* x^\delta$ have the same solutions. Setting

$$\begin{aligned} A_n &:= i_n^* i_n + \alpha I \ , && (3.23) \\ y_n &:= i_n^* x^\delta \ , && (3.24) \\ A &:= i^* i + \alpha I \ , && (3.25) \\ y &:= i^* x^\delta \ , && (3.26) \end{aligned}$$

$A, A_n : H^s(\Omega) \to H^s(\Omega)$, we obtain

$$\|A^{-1}\| = \|(i^* i + \alpha I)^{-1}\| \leq \frac{1}{\alpha} \tag{3.27}$$

and

$$\begin{aligned} \|(A - A_n)x\| &= \|(i^* i - i_n^* i_n)x\| \leq \|i\| \|(i - i_n)x\| + \|i^* - i_n^*\| \|i_n x\| \\ &= (\|i_n\| + \|i\|) \|i - i_n\| \|x\| \ . && (3.28) \end{aligned}$$

Now these operator norms can be estimated as follows:

$$\begin{aligned} \|(i_n - i)x\|_{L_2(\Omega)} &= \|(P_n - I)x\|_{L_2(\Omega)} \leq 2^{-sm(n)} \|x\|_{H^s(\Omega)} && (3.29) \\ \|i_n x\|_{L_2(\Omega)} &= \|P_n x\|_{L_2(\Omega)} \leq \|x\|_{H^s(\Omega)} && (3.30) \\ \|ix\|_{L_2(\Omega)} &\leq \|x\|_{H^s(\Omega)} \ , && (3.31) \end{aligned}$$

i.e.

$$\|A^{-1}(A - A_n)\| \le \frac{1}{\alpha} 2^{1-m(n)s} \ . \tag{3.32}$$

Altogether we conclude from (3.27)–(3.32),

$$\|(i^* - i_n^*)x^\delta\| \le \|i - i_n\| \|x^\delta\| \le 2^{-sm(n)} \|x^\delta\|$$

and (3.20) yields

$$\|x_{\alpha,n}^\delta - x_\alpha^\delta\| \;\; \le \;\; \frac{2^{-sm(n)}}{\alpha - 2^{1-sm(n)}} \left(2\|x_\alpha^\delta\| + \|x^\delta\| \right) \ . \tag{3.33}$$

(3.34)

From the minimizing property of x_α^δ follows

$$\alpha \|x_\alpha^\delta\|_{H^s}^2 \le J_\alpha(x_\alpha^\delta) \le J_\alpha(0) = \|x^\delta\|_{L_2}^2$$

and thus (3.22). ■

Formulas (3.21) and (3.22) show that the number of ansatz functions has to be adjusted to the regularization parameter α. In order to obtain proper convergence rates, n has to be choosen such that the right hand side is of order $O(\delta^{\nu/(\nu+1)})$, see Theorem 3.1.

3.3 Wavelet based computation of the discretized operator

In this section, we apply the results of Section 2.2 in order to solve the discretization problem described in Section 3.2. Moreover, we discuss a preconditioning of the proposed inversion scheme.

We start by relating to the subscript n some scale index j, i.e. $B_n = V_{j(n)}$. By $\Omega = [0,1)$, the dimension of V_j is determined by the cardinality of Φ_j. Consequently, for each scale index j there is a corresponding finite index n. Moreover, we may naturally identify Q_j as the orthogonal projection P_n. This leads to a reformulation of (3.18) by means of basis functions belonging to Φ_j

$$\bar{x}_{\alpha,j}^\delta = (E_{s,j}^{-1} E_j + \alpha I_j)^{-1} E_{s,j}^{-1} E_j x^\delta \ . \tag{3.35}$$

Moreover, to estimate the distance $x_{\alpha,j}^\delta - x^\delta$ we may apply the direct wavelet estimate (2.10). Then, we deduce from Proposition 3.3 the following

Corollary 3.1 *Let α be given, and assume that $2^{1-js} < \alpha$ holds (for some $s \geq 0$). Then the distance between x_α^δ and $x_{\alpha,j}^\delta$ can be estimated by*

$$\|x_{\alpha,j}^\delta - x_\alpha^\delta\| \leq \frac{2+\sqrt{\alpha}}{\sqrt{\alpha}} \frac{2^{-js}}{\alpha - 2^{1-js}} \|x^\delta\| \ . \tag{3.36}$$

In order to finally derive the approximation (3.35) we have to compute the entries of $E_{s,j}$ given by $\langle \phi_\lambda, \phi_{\lambda'} \rangle_{H^s(\Omega)}$, where $|\lambda| = |\lambda'| = j$. The finiteness of the entries is ensured if the chosen system Φ_j satisfies the H^s stability property, see Theorem 2.1. For an elegant computation of the entries we aim to apply Theorem 2.2. For this reason, we have to find a more suitable variant of inner products. Let us consider the space $\tilde{H}^s(\Omega) := \{f, \ |||f|||_s < \infty\}$ equipped with the norm $|||f|||_s^2 = \int (1+|\omega|^{2s})|\hat{f}(\omega)|^2 d\omega < \infty\}$. It can be shown that the spaces $\tilde{H}^s(\Omega)$ and $H^s(\Omega)$ coincide. The inner product of $\tilde{H}^s(\Omega)$ is of the form

$$\langle \phi_k, \phi_l \rangle_{\tilde{H}^s} = \langle \phi_k, \phi_l \rangle_{L_2} + \langle \phi_k, \phi_l^{(2s)} \rangle_{L_2} \ . \tag{3.37}$$

Note that by Theorem 2.2 we are restricted to $s \in \mathbb{N}/2$. To our knowledge there exist no uniqueness result including arbitrary fractional order of s. However, we may also solve the eigenvalue problem for some arbitrary s. But we are only done if the solved eigenvector (related to the eigenvalue 2^{2s}) is of simple order.

Combining all ingredients , we summarize the following nodal basis based numerical scheme for computing $\bar{x}_{\alpha,j}^\delta$:

1. *Choose with respect to the underlying measurement x^δ a suitable system Φ_j and an adequate resolution level j,*

2. *compute the matrix $C = (c_{2\alpha-\beta})_{\alpha,\beta}$ on the basis of the filter $\{h_\nu\}$,*

3. *solve the eigenvalue problems*

   $$(C - I_n)\mathcal{V}_1 = 0 \quad \text{and} \quad (C - 2^{-2s} I_n)\mathcal{V}_2 = 0$$

   for each column of $E_{s,j}$, where the components of $\mathcal{V}_1$ and $\mathcal{V}_2$ contain the entries of $E_{s,j}$ with respect to decomposition (3.37),

4. *solve*

   $$\bar{x}_{\alpha,j}^\delta = (E_{s,j}^{-1} E_j + \alpha I_j)^{-1} E_{s,j}^{-1} E_j \bar{x}^\delta = (I_j + \alpha E_{s,j})^{-1} \bar{x}^\delta \ .$$

   where the latter equation uses the fact that $E_j = I_j$.

When it comes to numerical realizations we should take into account that $E_{s,j}$ in the nodal basis Φ_j may exhibit a polynomial growth of the spectral condition number proportional to their size. The polynomial growth can be circumvented by wavelet preconditioning techniques, i.e. the nodal matrix $E_{s,j}$ must be transformed into a matrix relative to the multilevel basis

$$M_{s,j} := L_j^{-1} E_{s,j} L_j \ ,$$

where L_j denotes the related bi-orthogonal wavelet transform. It is known that $\mathrm{cond}(D_j^{-1} M_{s,j} D_j^{-1}) = \mathcal{O}(1)$ for $j \to \infty$ and D_j being a power 2 block diagonal matrix.

The application of wavelet preconditioning results in an equivalent system

$$y^\delta_{\alpha,j} = (D_j^{-2} + \alpha D_j^{-1} M_{s,j} D_j^{-1})^{-1} y^\delta \ ,$$

where $y^\delta_{\alpha,j} = D_j L_j^{-1} \bar{x}^\delta_{\alpha,j}$ and $y^\delta = D_j^{-1} L_j^{-1} x^\delta$. By stability arguments there is an adequate α such that

$$r_2(\alpha) := \alpha^{-1} \|(D_j^{-1} M_{s,j} D_j^{-1})^{-1}\| < 1$$

uniformly in j. Consequently, we have

$$\mathrm{cond}(\alpha^{-1} D_j^{-2} + D_j^{-1} M_{s,j} D_j^{-1}) < \frac{2}{1 - r_2(\alpha)} \,\mathrm{cond}(D_j^{-1} M_{s,j} D_j^{-1}) \ .$$

The upper bound of the latter estimate can be made small if α was chosen adequately large. This requirement on α does not contradict to the lower bound assumption on α made in Corollary 3.1.

3.4 Regularization by Fourier series

In this section we aim at solving our problem by discrete Fourier series, which is of particular interest if the FFT is used. Hence, we have to switch to a periodic setting. We assume that x is a periodic function over the interval $\Omega = [0, 2\pi)$. Consequently, x can be expressed by

$$x(t) = \sum_k x_k e^{ikt} \ , \tag{3.38}$$

where $t \in \mathbb{R}$ and x_k denotes the Fourier coefficients. By (3.38) we obtain for $x, y \in L_2(\Omega)$ and $x, y \in H^s(\Omega)$ respectively that

$$\langle x, y\rangle_2 = \sum_k x_k y_k \quad \text{and} \quad \langle x, y\rangle_s = \sum_k (1 + k^2)^s x_k y_k \tag{3.39}$$

respectively. Again, we assume that x is a given signal with $x \in H^s(\Omega)$ and the compact mapping $i : H^s(\Omega) \to L_2(\Omega)$ is given for $x \in H^s(\Omega)$ by $i(x) = x$.

As in Section 3.1 we are going to characterize the range of the operator $(i^*i)^{\nu/2}$ and will give an explicit formula for the computation of the minimizer of the Tikhonov–functional.

Lemma 3.3 *Assume that for some $x \in H^s(\Omega)$ the compact embedding*

$$i : H^s(\Omega) \to L_2(\Omega) \quad \textit{is given by} \quad i(x) = x \ .$$

Then, the minimizing element x_α^δ of (3.1) is given by

$$x_\alpha^\delta = \sum_k \left(x_\alpha^\delta\right)_k e^{ikt} \ , \tag{3.40}$$

$$\left(x_\alpha^\delta\right)_k = \frac{\left(x^\delta\right)_k}{1+\alpha(1+k^2)^s} \ . \tag{3.41}$$

Proof We only have to prove assertion (3.40). At first, we verify for $iy, x \in L_2(\Omega)$ that

$$\begin{aligned} \langle iy, x\rangle_2 &= \sum_k y_k x_k = \langle y, i^*x\rangle_s \\ &= \sum_k (1+k^2)^s y_k (i^*x)_k \ . \end{aligned}$$

Consequently, we have for all $y \in H^s(\Omega)$ that

$$0 = \sum_k y_k \left(x_k - (i^*x)_k(1+k^2)^s\right) = \langle y, z\rangle_s \ .$$

Thus, $z = 0$, where

$$z_k = x_k(1+k^2)^{-s} - (i^*x)_k \ ,$$

i.e.,

$$(i^*x)_k = x_k(1+k^2)^{-s} \ . \tag{3.42}$$

The minimizing function $x_\alpha^\delta \in H^s(\Omega)$ of (3.1) is a solution of the equation

$$(i^*i + \alpha i)x = i^*x^\delta \ , \tag{3.43}$$

cf. (2.7), and has a representation (3.40). We obtain by using (3.42)

$$\begin{aligned}\sum_k (x_\alpha^\delta)_k(\alpha + (1+k^2)^{-s})e^{ikt} &= (i^*i+\alpha)x_\alpha^\delta(t)\\ &= i^*x^\delta(t)\\ &= \sum_k (x^\delta)_k(1+k^2)^{-s}e^{ikt}\ .\end{aligned}$$

Hence, we can conclude that

$$(x_\alpha^\delta)_k = \frac{(x^\delta)_k}{1+\alpha(1+k^2)^s}\ .$$

■

Thus we have a simple formula for the computation of x_α^δ. If the Fast Fourier Transform is used for the computation of the Fourier coefficients of x^δ, then the computation is additionally fast.

Next we want to characterize the range condition (2.4) into a smoothness conditions in terms of Sobolev spaces, which is much easier to check. For this reason we need the Singular Value Decomposition of i^*i:

Lemma 3.4 *Let $v_k = e^{ik\cdot}$, $u_k = (1+k^2)^{-s/2}e^{ik\cdot}$, and $\sigma_k = (1+k^2)^{-s/2}$. Then $\{\sigma_k^2, u_k\}$ is the eigensystem of i^*i and $\{\sigma_k, u_k, v_k\}$ forms the singular system of i.*

Proof By the definition of u_k, v_k and (3.39) holds

$$\begin{aligned}\langle x, u_k\rangle_s &= (1+k^2)^{-s/2}\langle x, v_k\rangle_s\\ &= x_k(1+k^2)^{-s/2}(1+k^2)^s\ .\end{aligned}\tag{3.44}$$

Moreover, we have

$$\begin{aligned}i^*ix &= i^*x\\ &= \sum_k (i^*)_k v_k\\ &\overset{(3.42)}{=} \sum_k x_k(1+k^2)^{-s}v_k\\ &= \sum_k x_k(1+k^2)^{-s/2}u_k\\ &\overset{(3.44)}{=} \sum_k (1+k^2)^{-s}\langle x, u_k\rangle u_k\ ,\end{aligned}$$

and thus $\{\sigma_k^2, u_k\}$ is the eigensystem of (i^*i). For the embedding operator i it is easily seen that for $x \in H^s$ holds

$$\begin{aligned} ix &= x = \sum_k \langle x, v_k\rangle v_k \\ &= \sum_k (1+k^2)^{-s}\langle x, v_k\rangle_s v_k \\ &= \sum_k \sigma_k \langle x, u_k\rangle_s v_k \ , \end{aligned}$$

and $\{\sigma_k, u_k, v_k\}$ is the singular system of i. ■

Lemma 3.5 *Let the compact embedding $i : H^s(\Omega) \to L_2(\Omega)$ be given by $i(x) = x$. Then $x = (i^*i)^{\nu/2}w$ holds if and only if $x \in H^{s+\nu s}$.*

Proof With $w \in H^s(\Omega)$ we obtain for all $z \in H^s(\Omega)$

$$\begin{aligned} \sum_k z_k x_k (1+k^2)^s &= \langle z, x\rangle_s = \langle z, (i^*i)^{\nu/2}w\rangle_s \\ &= \langle z, \sum_k \sigma_k^\nu \langle w, u_k\rangle_s u_k\rangle_s \quad . \\ &\overset{(3.44}{=} \sum_k \sigma_k^\nu w_k z_k (1+k^2)^s \end{aligned} \tag{3.45}$$

Hence, we have

$$x_k(1+k^2)^{s\nu/2} = w_k \ ,$$

and because of $w \in H^s(\Omega)$ we conclude

$$\|w\|_s^2 = \sum_k (1+k^2)^s w_k^2 = x_k^2(1+k^2)^{s\nu}(1+k^2)^s < \infty \tag{3.46}$$

which is equivalent to $x \in H^{s+s\nu}(\Omega)$. ■

We might summarize our results in the following

Theorem 3.2 *Let the compact embedding $i : H^s(\Omega) \to L_2(\Omega)$ be given by $i(x) = x$, x^δ be a noisy version of $x \in H^s(\Omega)$ with $\|x - x^\delta\|_0 \le \delta$, and*

x_α^δ *be the minimizing element of the Tikhonov–functional (3.1), where the regularization parameter is determined via Morozov's discrepancy principle. If* $x \in H^{s+s\nu}$*, then the error estimate*

$$\|x_\alpha^\delta - x\|_s = O(\delta^{\nu/(\nu+1)}) \tag{3.47}$$

holds.

Acknowledgement

The authors gratefully acknowledge support by DFG grants Te 354/1-2 and Ma 1657/7-2.

References

[1] G. Bruckner, *On the regularization of the ill-posed logarithmic kernel integral equation of the first kind* Inverse Problems, **11**, 65-77, 1995.

[2] G. Bruckner, *On the stabilization of approximation procedures for a class of integral equations of the first kind with noisy right hand sides* J. Inv. and Ill-Posed Problems **5**, 117-127, 1997.

[3] W. Dahmen, *Some remarks on multiscale transformations, stability and biorthogonality*, in: Curves and Surfaces II, P. J. Laurent, A. Le Méhauté, and L. L. Schumaker (eds.), AKPeters Wellesley, Boston, 157-188, 1994.

[4] W. Dahmen, *Stability of multiscale transformations*, The Journal of Fourier Analysis and Applications, **2**, 341–361, 1996.

[5] W. Dahmen, A. Kunoth, *Multilevel Preconditioning*, Numerische Mathematik, **(63)**, 315-344, 1992.

[6] W. Dahmen, Ch. Micchelli, *Using refinement equations for evaluating integrals of wavelets*, SIAM Journal on Numerical Analysis, **30(2)**, 507-537, 1993.

[7] I. Daubechies ,*Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[8] I. Daubechies, *Wavelet Transforms and Orthonormal Wavelet Bases*, Proceedings of Symposia in Applied Mathematics, **47**, 1993.

[9] D. Donoho, I.M. Johnstone, *Minimax estimation via wavelet shrinkage*, Biometrika ,**81**, 425-455, 1994.

[10] H. W. Engl, M. Hanke, A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1996.

[11] A. Frommer, P. Maass, *Fast CG-based methods for Tikhonov regularization*, SIAM J. Sci. Comp., **5(20)**, 1831-1850, 1999.

[12] C.W. Groetsch and O. Scherzer, *Nonstationary iterated Tikhonov-Morozov method and third order differential equations for the evaluation of unbounded operators* Math. Meth. Appl. Sci. **23**, 1287-1300, 2000.

[13] R. Kress, *Linear Integral Equations*, Springer, New York, 1989.

[14] V. Lehmann, G. Teschke, *Wavelet based methods for improved wind profiler signal processing*, Annales Geophysicae, **19**, 825-836, 2001.

[15] A. K. Louis, *Inverse und schlecht gestellte Probleme*, Teubner, Stuttgart, 1989.

[16] P. Maass, *Wavelet Methods in Signal Processing* , Mathematical Methods in Metrology , ed. H. Lübbig , Berlin , 1996.

[17] A. K. Louis, P. Maass, A. Rieder, *Wavelets*, Teubner, Stuttgart, 1998.

[18] P. Maass, S. V. Pereverzev, R. Ramlau, S. G. Solodky, *An adaptive discretization scheme for Tikhonov–regularization with a posteriori parameter selection*, Numerische Mathematik, **3(87)**, 485–502, 2001.

[19] F. Natterer, *Sobolev space analysis of picture reconstruction*, SIAM J. Appl. Math., **39**, 402-411, 1980.

[20] F. Natterer, *The mathematics of Computerized Tomography*, Teubner & Wiley, Stuttgart, 1986.

[21] S. Osher, A. Solé, L. Vese, *Image decomposition and restoration using total variation minimization and the H^{-1} norm*, Report **02-57**, University of California Los Angeles C.A.M., 2002.

[22] R. Ramlau and G. Teschke, *Regularization of Sobolev Embedding Operators and Applications Part II: Data Driven Regularization and Applications*, to appear in: Sampling Theory in Signal and Image Processing, 2004.

[23] J. Weikert, *Anisotropic Diffusion in Image Processing*, Teubner, Stuttgart, 1998.