A Wavelet Based Numerical Method for Nonlinear Partial Differential Equations

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Abstract

The purpose of this paper is to present a wavelet–Galerkin scheme for solving nonlinear elliptic partial differential equations. We select as trial spaces a nested sequence of spaces from an appropriate biorthogonal multiscale analysis. This gives rise to a nonlinear discretized system. To overcome the problems of nonlinearity, we apply the machinery of interpolating wavelets to obtain *knot oriented* quadrature rules. Finally, Newton's method is applied to approximate the solution in the given ansatz space. The results of some numerical experiments with different biorthogonal systems, confirming the applicability of our scheme, are presented.

1 Introduction

In recent years, the interest on using wavelets as a tool for the numerical treatment of partial differential equations has been growing considerably. So far, the most far-reaching results were obtained for linear, boundedly invertible operator equations; see, e.g. [4, 3, 5, 6]. (This list is clearly not complete.) Once these problems are well–understood and almost completely solved, the next challenging task is the numerical treatment of *nonlinear* problems. This paper can be viewed as one small step in this direction. We shall be concerned with the numerical treatment of nonlinear partial differential equations of the type

$$F(u) := Lu + G(u) = 0, \tag{1}$$

on some bounded domain $\Omega \subset \mathbb{R}^d$, where L is an elliptic linear differential operator of second order and G is a nonlinear operator.

As a typical example we will focus here on the model problem

$$Lu = -\Delta u + u \quad \text{and} \quad G(u) = u^3 - f, \tag{2}$$

for a given function $f \in L^2(\Omega)$.

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To simplify the description of the method and to avoid possible boundary effects, we will concentrate on the 1D case and on periodic problems, i.e. we will take Ω to be the unidimensional torus $\Omega = \mathcal{T} := \mathbb{R}/\mathbb{Z}$. This might sound artificial at first sight. However, one of our aims is to apply our scheme to problems in image processing where the use of periodic boundary conditions is quite standard. Also, the generalization to problems in higher dimensions, via tensor product techniques, is straightforward.

By modifying in an obvious manner the analysis contained in [1], one can show that the above problem has a unique solution, which we now aim to approximate numerically. In order to do this, we first consider the associated weak formulation

$$a(u, v) + \int_{\mathcal{T}} g(u)v \, dx - \int_{\mathcal{T}} f v dx = 0 \,, \qquad \text{for all } v \in H^1 \,, \tag{3}$$

where $a(.,.)$ is the bilinear form induced by L and the prescribed boundary conditions

$$a(u, v) = \int_{\mathcal{T}} \nabla u \cdot \nabla v \, dx + \int_{\mathcal{T}} uv \, dx. \tag{4}$$

Here, $H^1 := H^1(\mathcal{T})$ denotes the first order Sobolev space on $\mathcal{T}$ equipped with the usual norm $\| \cdot \|_1$ and $g(u) = u^3$. We can then use a classical Galerkin approach, i.e., consider a nested sequence of finite–dimensional approximation spaces $\{V_j\}_{j \geq 0}$, $V_j \subset H^1$, and look for an approximation $u_j \in V_j$ to the solution u of the problem by solving

$$a(u_j, v_j) + \int_{\mathcal{T}} g(u_j)v_j \, dx - \int_{\mathcal{T}} f v_j \, dx = 0 \,, \qquad \text{for all } v_j \in V_j. \tag{5}$$

2 Multiresolution Analysis

Our goal is to derive efficient Galerkin methods for the approximate solution of (3). Here, in contrast to conventional finite element discretizations, we will work with trial spaces that, not only exhibit the usual approximation and good localization properties, but in addition lead to *expansions* of any element in the underlying Hilbert spaces in terms of *multiscale* or *wavelet* bases with certain stability properties.

We now briefly review some essential features of wavelet approximation that will be important in the sequel; for more details the reader is referred to, e.g. [2]. We start by recalling the concept of multiresolution. A multiresolution analysis (MRA) (V_j, ϕ) of $L_2(\mathbb{R})$ is a sequence of closed subspaces of $L_2(\mathbb{R})$ and an associated function ϕ, called the *generator* or *scaling function*, satisfying:

$$V_j \subset V_{j+1}, \quad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}), \quad \bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \tag{6}$$

$$f \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1}, \tag{7}$$

$$\{\phi(\cdot - k) : k \in \mathbb{Z}\} \text{ is a Riesz basis of } V_0. \tag{8}$$

It then follows that, for each j, the set of functions $\{\phi_{j,k} := 2^{j/2}\phi(2^j. - k) : k \in \mathbb{Z}\}$ is a Riesz basis for the space V_j (the so-called nodal basis). Wavelets are associated with *detail spaces*, i.e. with complementary spaces W_j satisfying $V_{j+1} = V_j \oplus W_j$, where $\oplus$ denotes a direct sum. The properties of a multiresolution analysis imply that $\bigoplus_{j\in\mathbb{Z}} W_j = L_2(\mathbb{R})$. Hence, if we can find a function ψ whose integer translates form a Riesz basis of W_0, then the collection $\{\psi_{j,k} := 2^{j/2}\psi(2^j \cdot -k) : j,\ k \in \mathbb{Z}\}$ will be a good candidate for a Riesz basis for the space $L_2(\mathbb{R})$ (a so-called *wavelet basis*). Of course, there is a continuum of possible choices of such complement spaces. Orthogonal decompositions would lead to orthogonal wavelets. However, orthogonality often interferes with locality and the actual computation of orthonormal bases might be too expensive. Moreover, in certain applications orthogonal decompositions are actually not best possible. These limitations motivated the search for biorthogonal wavelet bases, i.e., for bases $\{\psi_{j,k}\}$ and $\{\widetilde{\psi}_{j,k}\}$ which satisfy

$$\left\langle \psi_{j,k}, \widetilde{\psi}_{j',k'} \right\rangle = \delta_{j,j'}\delta_{k,k'}, \quad \text{for all } j, j', k, k' \in \mathbb{Z}, \tag{9}$$

with $\langle \cdot, \cdot \rangle$ denoting the usual inner product in $L_2(\mathbb{R})$. These biorthogonal wavelet bases can be associated with two multiresolution analyses (V_j, ϕ) and $(\widetilde{V}_j, \widetilde{\phi})$ which are connected in the following manner:

$$L_2(\mathbb{R}) = V_0 \oplus (\widetilde{V}_0)^{\perp}, \tag{10}$$

where $X^{\perp}$ denotes the orthogonal complement of X in $L_2(\mathbb{R})$. Two such multiresolution analyses are said to form a *dual* or *biorthogonal* multiresolution analysis of $L_2(\mathbb{R})$. In this case, we can define the detail spaces W_j and $\widetilde{W}_j$ by $W_j = V_{j+1} \cap \widetilde{V}_j^{\perp}$ and $\widetilde{W}_j = \widetilde{V}_{j+1} \cap V_j^{\perp}$. Each function $v_j \in V_j$ will have a nodal basis representation representation, as

$$v_j = \sum_k c_{j,k}\, \phi_{j,k} \tag{11}$$

and also a so-called *multilevel* representation

$$v_j = \sum_k c_{j_0,k}\phi_{j_0,k} + \sum_{j\geq j_0}\sum_k d_{j,k}\psi_{j,k}, \tag{12}$$

where j_0 denotes a certain chosen coarsest level and $c_{j,k} = \left\langle v_j, \widetilde{\phi}_{j,k} \right\rangle$ and $d_{j,k} = \left\langle v_j, \widetilde{\psi}_{j,k} \right\rangle$. The results above where given for spaces defined on the whole of $\mathbb{R}$. To adapt this construction to the periodic setting, we can simply use a well-known procedure of periodization, as described in detail in, e.g., [2]. For a given pair of dual multiresolution analyses of $L_2(\mathbb{R})$, with dual scaling functions ϕ and $\widetilde{\phi}$, we can consider the periodized functions

$$[\phi_{j,k}] = 2^{j/2} \sum_{l\in\mathbb{Z}} \phi(2^j(\cdot + l) - k)$$

and define the spaces

$$[V_j] := \operatorname{Span}(\{[\phi_{j,k}] :\ k = 0, 1 \ldots, 2^j - 1\}), \quad j \in \mathbb{N}_0.$$

(Similar constructions, naturally, associated with $\widetilde{\phi}$.) Then, it can easily be shown that these finite dimensional spaces are nested and dense in $L_2(\mathcal{T})$. If ψ and $\widetilde{\psi}$ are the basic wavelets associated with the biorthogonal multiresolution analyses and we consider also the periodized functions $[\psi_{j,k}]$ and $[\widetilde{\psi}_{j,k}]$, $j \in \mathbb{N}_0, k = 0, 1 \ldots, 2^j - 1$, then, it can be shown that the orthogonality conditions (9) carry over to analogous conditions relative to the inner product defined on $L_2(\mathcal{T})$.

In this paper, we propose to adopt for test spaces in the Galerkin method, the (periodized) spaces $[V_j]$ of a dual multiresolution analysis generated by scaling functions selected from the family of the so called *Deslauriers-Dubuc fundamental functions*. These functions, which are obtained as auto-correlation of the well-known compact support orthogonal Daubechies scaling functions, have very attractive properties, particularly important for our purposes; see, e.g., [10, 9]. In particular, each of these functions is compactly supported and is also *interpolating*, i.e., satisfies $\phi(k) = \delta_{0k}$, $k \in \mathbb{Z}$. Also, several dual functions $\widetilde{\phi}$ to ϕ, with certain smoothness properties, can be obtained; see. e.g. [11] and [8] for details.

Recalling the Galerkin approach (5) to solve our problem, we see that we obtain the following system for determining the coefficient vector $\mathbf{c}_j := (c_{j,k})$ of the approximate solution $u_j \in V_j$ in the nodal basis

$$\sum_{k\in\Lambda_j} c_{j,k} a([\phi_{j,k}],[\phi_{j,k'}]) + \int_{\mathcal{T}} g(\sum_{k\in\Lambda_j} c_{j,k}[\phi_{j,k}])[\phi_{j,k'}]dx - \int_{\mathcal{T}} f[\phi_{j,k'}]dx = 0 \qquad k' \in \Lambda_j,$$

where $\Lambda_j := \{0, 1, \ldots, 2^{j-1}\}$. Due to the nonlinear structure of the function g, we are faced with the problem of adequately estimating the integral $\int_{\mathcal{T}} g(\sum_{k\in\Lambda_j} c_{j,k}[\phi_{j,k}])[\phi_{j,k'}]dx$. In this paper, we suggest to approximate the above integral by a version of the so called *knot oriented* quadrature rules used in the finite element setting. We replace $\int_{\mathcal{T}} f(x)dx$ by $\int_{\mathcal{T}}(I_j f)(x)dx$, where I_j is an *interpolation projector* induced by the interpolating scaling function ϕ and defined as follows:

$$(I_j f) := \sum_k f(2^{-j}k)\phi(2^j \cdot -k). \tag{13}$$

Some properties of this interpolation operator can be seen in [7]. We thus obtain the following approximation

$$\int_{\mathcal{T}} g\left(\sum_{\lambda\in\Lambda_j} c_{j,k}[\phi_{j,k}]\right)[\phi_{j,k'}]dx \approx 2^{-j/2} g(2^{j/2} c_{j,k'}).$$

This leads to a modified Galerkin system, which can be written as

$$\mathbf{A}_j \cdot \mathbf{c}_j + \widetilde{\mathbf{g}}_j(\mathbf{c}_j) - \mathbf{f} = 0. \tag{14}$$

Here, we propose to use of Newton's method for solving the above system. With $J = 2^j$, let $\mathbf{F}_j : \mathbb{R}^J \to \mathbb{R}^J$ be the mapping defined by

$$\mathbf{F}_j(\xi) := \mathbf{A}_j\ \xi + \widetilde{\mathbf{g}}_j(\xi) - \mathbf{f}. \tag{15}$$

Then, we apply the following iterative scheme, starting with an appropriate initial vector $\mathbf{d}_j^{(0)}$:

$$\begin{cases} \mathbf{F}_j'(\mathbf{d}_j^{(n)})\ \zeta^{(n+1)} = \mathbf{F}_j(\mathbf{d}_j^{(n)}) \\ \mathbf{d}_j^{(n+1)} = \mathbf{d}_j^{(n)} - \zeta^{(n+1)}. \end{cases} \tag{16}$$

The Jacobian matrix of $\mathbf{F}_j$ is naturally given by

$$\mathbf{F}_j'(\xi) = \mathbf{A}_j + \begin{pmatrix} g'(2^{j/2}\xi_1) & & 0 \\ & \ddots & \\ 0 & & g'(2^{j/2}\xi_J) \end{pmatrix} \tag{17}$$

In every Newton step one has to solve a linear system of equations. Hence, we have to be careful about the condition of $\mathbf{F}_j'$. A detailed discussion of this problem is given in [7]. We just refer here that it is necessary to use a wavelet preconditioner that realizes a change of basis from nodal to multilevel; the numerical results presented were obtained by doing this change of basis.

3 Numerical Results

To confirm the quality of our approach we present some test examples that describe e.g. a chemical intermixture process. Numerical results were obtained by choosing the (periodized) Deslauriers-Dubuc interpolating scaling functions ϕ_{2N}; $N = 2, 3, 4$, with a dual $\widetilde{\phi}_{2N,1}$, whose construction is described in detail in [11] and [8]. We assume that the function u is given by $u(x) = \sin(2\pi x)$ which is a sufficiently smooth periodic function on $\Omega = [0, 1]$. Then, we specify f as $f(x) = \sin^3(2\pi x) + ((2\pi)^2 + 1)\sin(2\pi x)$. The numerical approximations are displayed in the figure below.

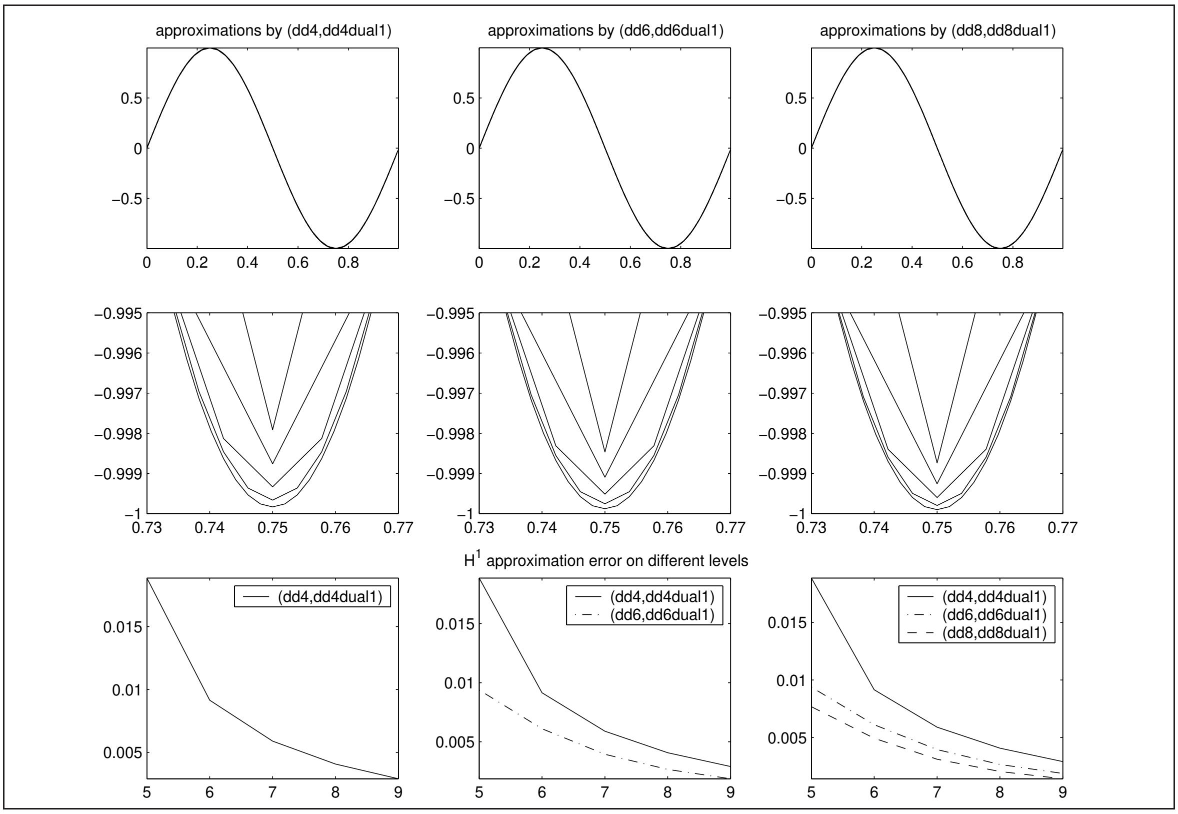

Figure 1: Approximations on scales $j = 5$ through $j = 9$ with trial functions of varying smoothness and H^1-approximation error (below).

Remark : A more thorough investigation of the method here proposed, including a detailed analysis of convergence and more numerical results are given in a forthcoming paper; see [7].

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