A partial differential equation for continuous non-linear shrinkage filtering and its application for analyzing MMG data

Kristian Bredies, Dirk A. Lorenz, Peter Maass, Gerd Teschke

Zentrum für Technomathematik, University of Bremen, Bibliothekstr. 1, 28359 Bremen, Germany

ABSTRACT

The starting point for this paper is the well known equivalence between convolution filtering with a rescaled Gaussian and the solution of the heat equation. In the first sections we analyze the equivalence between multi-scale convolution filtering, linear smoothing methods based on continuous wavelet transforms and the solutions of linear diffusion equations. I.e. we determine a wavelet ψ, resp. a convolution filter φ, which is associated with a given linear diffusion equation $u_t = Pu$ and vice versa.

This approach has an extension to non-linear smoothing techniques. The main result of this paper is the derivation of a differential equation, whose solution is equivalent to non-linear multi-scale smoothing based on soft shrinkage methods applied to Fourier or continuous wavelet transforms.

Keywords: image smoothing, multi-scale methods, wavelet transform, diffusion equations

1. INTRODUCTION

Several different concepts for non-linear filtering methods have been introduced in the last decade. They all aim at preserving or even enhancing specific structure while at the same time reducing the influence of noise.

Non-linear evolution equations and wavelet-based shrinkage methods are amongst the most prominent examples of this kind. They have been successfully applied to a large variety of real life problems in signal and image processing.

In this paper we will analyze the equivalence between these concepts. At first we show some equivalences between convolution filters, linear wavelet filters and pseudo differential equations. Later we will consider shrinkage filtering applied to the continuous wavelet or Fourier transform of a function u_0 and derive an evolution equation

$$u_t + \partial\Phi(u) \ni 0\ , \quad u(0,\cdot) = u_0\ ,$$

whose solution $u(t,x)$ is equivalent to applying a shrinkage method with threshold t. I.e. the appropriate choice of a functional Φ allows to define a descend method along a subgradient of Φ, which yields a shrinkage filter. In all cases considered, this functional is based on the L_1-norm of either the wavelet or the Fourier transform of u.

In the last section, we will present the results of an application of wavelet shrinkage methods to magneto-myographic (MMG) data.

2. CONVOLUTION AND LINEAR WAVELET FILTERING

In this section we collect some basic results, which highlight the relations between linear convolution filters and equivalent (pseudo-)differential operators.

A function $\varphi \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\ dx = 1$ is called a convolution filter. $\varphi(x,t) = \frac{1}{t^n}\varphi(x/t)$ denotes the convolution filter at scale t.

For a function $u_0 \in L_2(\mathbb{R}^n)$ we define its multi-scale filtered version by

$$u(x,t) = (u_0 * \varphi(\cdot,t))\,(x)\ .$$

For a given convolution filter φ we have the well known result, that under some assumptions we have a corresponding pseudo differential operator P so that the solution of the initial value problem

$$\frac{\partial u}{\partial t} = Pu,\ \ u(\cdot,0) = u_0$$

is given by

$$u(x,t) = \Big(u_0 * \varphi(\cdot,t)\Big)(x)\ .$$

The pseudo differential operator P has the symbol

$$\sigma(t,\omega) = \frac{\partial}{\partial t}\Big(\ln(\hat{\varphi}(t\omega))\Big)$$

where $\hat{\varphi}$ denotes the Fourier transform of φ.

The assumptions that have to be made on φ say something about the differentiability and the growth properties of $\hat{\varphi}$. Details can be found in.[1]

Example. However, not every linear convolution filters has an equivalent representation by a differential equation of type $u_t = Pu$. Even the most common case of the moving average filter, i.e. the convolution filter $\varphi(x) = \chi_{[-1/2,1/2]}(x)$ with $\hat{\varphi}(\omega) = \mathrm{sinc}(\omega)$, does not fall in this class.

The original signal is again denoted by u_0. A direct computation of $\frac{\partial}{\partial t}u(t,x)$ yields ($t > 0$):

$$\frac{\partial}{\partial t}\hat{u}(t,\omega) \ \ = \ \ \frac{1}{t}\left\{-\hat{u}(t,\omega) + \cos(t\omega)\hat{u}_0(\omega)\right\}\ .$$

Hence the linear convolution yields a function $u(t,x)$, which is a solution of the following initial value problem:

$$u_t = -\frac{1}{t}u \ + \ \frac{1}{2t}(u_0(x+t) + u_0(x-t)) \quad , \quad u(0,\cdot) = u_0 \quad .$$

A generalization of this approach leads to convolution filters which are equivalent to differential equations of type $u_t = Pu + Qu_0$.

The answer of the opposite question, namely which linear differential operators P with Fourier symbols σ allows an equivalent representation by a linear convolution filter, is almost similar:

For a given pseudo differential operator P with symbol $\sigma(t,\omega)$ we have a convolution filter φ, so that the equation $u_t = Pu$ is solved by $u(t,x) = (u_0 * \varphi(t,\cdot))(x)$. The convolution filter φ is given by

$$\hat{\varphi}(\omega) = \frac{1}{(2\pi)^{n/2}}\exp\psi(\omega) \ \ \text{with} \ \ \psi(\omega) = \int_0^1 \sigma(t,\omega)\ dt\ .$$

Surprisingly this result lacks of symmetry: The assumptions that have to be made on the symbol of the pseudo differential operator P to get a convolution filter φ are different from the properties of the symbol σ that we get, when we start from a convolution filter φ. Again the details can be found in.[1]

Another class of linear filters is given by cropping in the wavelet domain. The *wavelet transform* of a function $f \in L^2(\mathbb{R})$ with respect to a wavelet ψ is defined as

$$L_\psi f(a,b) = \frac{1}{\sqrt{c_\psi}}\int_{\mathbb{R}} f(x)|a|^{-1/2}\overline{\psi}\left(\frac{x-b}{a}\right)\ dx\ .$$

This transform is an isometry from $L^2(\mathbb{R})$ to $\operatorname{range} L_\psi \subset L^2(\mathbb{R}^2, \frac{dbda}{a^2})$. The inversion formula for $g \in \operatorname{range} L_\psi$ is thus given by the adjoint operator, which can be written as

$$L_\psi^* g(x) = \frac{1}{\sqrt{c_\psi}} \int_{\mathbb{R}} \int_{\mathbb{R}} g(a,b)|a|^{-1/2} \psi\left(\frac{x-b}{a}\right) \frac{dbda}{a^2} \ .$$

For a fixed $a > 0$ the details of f on scale a are represented by the functions $L_\psi f(a,\cdot)$ and $L_\psi f(-a,\cdot)$. Therefore, one obtains a multi-scale filtered version $u(t,x)$ of $f(x)$ by deleting the details smaller than t, or to be more precise:

$$u(t,x) = \frac{1}{\sqrt{c_\psi}} \int_{|a| \geq t} \int_{\mathbb{R}} L_\psi f(a,b)|a|^{-1/2} \psi\left(\frac{x-b}{a}\right) \frac{dbda}{a^2} \quad ; \quad t \geq 0$$

As before we want to analyze the question, whether there is also an equivalent differential equation, $u_t = Pu$, whose solution coincides with the wavelet filtering.

At first we have the result, that wavelet filtering is equivalent to a special choice of convolution filtering:

For a real valued wavelet ψ, the linear wavelet filtering is equivalent to a convolution filtering with the filter φ given by

$$\hat{\varphi}(\omega) = \frac{2\pi}{c_\psi} \int\limits_{|\xi| \geq |\omega|} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} \, d\xi \ .$$

Remark 1. *We can derive an equivalent wavelet filter to a given convolution filter by solving the equation*

$$|\omega| \frac{d}{d\omega} \hat{\varphi}(\omega) = \frac{2\pi}{c_\psi} |\hat{\psi}(\omega)|^2 \ .$$

Now the connection to a differential equation is easy: The symbol of the pseudo differential operator is given by $\sigma(t,\omega) = \frac{2\pi}{c_\psi} \frac{\frac{\partial}{\partial t}\hat{\varphi}(t\omega)}{\hat{\varphi}(t\omega)}$.

Here we have to take care of dividing by zero: We set $M_\psi = \sup\{w \mid |\hat{\psi}(\omega)| > 0\}$. Then we have

$$\hat{\varphi}(t\omega) \ = \ \frac{2\pi}{c_\psi} \int_{|a| \geq t} \frac{|\hat{\psi}(a\omega)|^2}{|a|} \, da \ = \ \frac{2\pi}{c_\psi} \int_{|a| \geq |t\omega|} \frac{|\hat{\psi}(a)|^2}{|a|} \, da \ > 0$$

for $|t\omega| < M_\psi$ as well as $\hat{\varphi}(\omega) = 0$ for $|\omega| \geq M_\psi$. So our definition of σ makes sense for $|t\omega| < M_\psi$ with an extension by zero for $|t\omega| \geq M_\psi$.

Again the answer of opposite question, i. e. for which symbol σ we have a representation of the smoothing by a wavelet filter with wavelet ψ, has the same asymmetry as in the convolution filtering case. Once again, the details are in.[1]

3. NONLINEAR METHODS

In this section we analyze non-linear filtering concepts based on continuous shrinkage methods. Shrinkage methods have been used for noise reduction in signal processing applications for quite some time already. However, a thorough mathematical analysis of shrinkage methods for discrete wavelet transforms has first been presented in.[2,3] These papers sparked a growing interest in shrinkage methods and lead to various generalizations.[4–14]

In this section we want to analyze shrinkage methods for continuous Fourier and continuous wavelet transform. Both methods operate in a three step procedure: the signal f is transformed; the modulus of the transformed signal is reduced by a value t; the inverse transform yields $u(t,x)$ with $u(0,x) = f(x)$, where t acts as scale or time parameter. The mathematical formulation of the Fourier shrinkage reads as

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\operatorname{sgn}(\hat{f}(\omega))[|\hat{f}(\omega)| - t]_+ \right) e^{i\omega x} d\omega \tag{1}$$

and the wavelet shrinkage:

$$u(t,x) = \frac{1}{\sqrt{c_\psi}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\operatorname{sgn}(L_\psi f(a,b))[|L_\psi f(a,b)| - t]_+ \right) |a|^{-1/2} \psi \left(\frac{x-b}{a} \right) \frac{dbda}{a^2}. \tag{2}$$

Our main result in this section will state equivalences between these non-linear multi-scale smoothing methods and differential equations. This work was inspired by,[15, 16] where a differential formulation for discrete wavelet shrinkage was derived.

Chambolle, Lucier, et al. showed in[15, 16] that the discrete wavelet shrinkage i. e.

$$u(t,x) = \sum_{\lambda \in \Lambda} \operatorname{sgn}(\langle f | \psi_\lambda \rangle)[| \langle f | \psi_\lambda \rangle | - t]_+ \psi_\lambda$$

corresponds to the solution of the following subgradient descent problem:

$$u_t + \partial\Phi(u) \ni 0, \; u(0) = f \quad \text{with} \quad \Phi(u) = \sum_{\lambda \in \Lambda} | \langle u | \psi_\lambda \rangle | \; .$$

We are going to show, that a similar statement is true for other orthogonal transforms like the Fourier transform. The case of the continuous wavelet transform is more involved.
Descent equations like this may look unfamiliar because of the "$\in$" instead of an "$=$". This is because the functional Φ is not differentiable and so we have to use the subdifferential, which is in general multivalued. We give definitions of these terms below.

The *domain* of a convex mapping $\Phi : H \to]-\infty, \infty]$ on a Hilbert space H is the set where Φ has finite values. It is *proper*, if $\operatorname{dom} \Phi \neq \emptyset$. Further it is said to be *lower semi-continuous* (lsc), if for every $a \in \mathbb{R}$ the set $\{f \in H \mid \Phi(f) \leq a\}$ is closed in H.

The *subdifferential* $\partial\Phi(f)$ at a point $f \in H$ consist of the *subgradients* $g \in H$, defined by the subgradient inequality

$$\forall h \in H \; : \; \Phi(h) \geq \Phi(f) + \langle g | h - f \rangle_H \; .$$

The domain of the subdifferential $\operatorname{dom} \partial\Phi$ is the set of functions f where the subdifferential is not empty.

It is a well known result from the theory of semi-groups, that the subgradient descent problem

$$\frac{\partial}{\partial t} u + \partial\Phi(u) \ni 0 \; ; \; u(0) = f$$

has a solution if Φ is proper, convex and lsc and if $f \in \overline{\operatorname{dom} \partial\Phi}$ (see[17]).

The key to our result is the fact that the soft shrinkage function $S_t(x_0) = \operatorname{sgn}(x_0)(|x_0| - t)_+$ is the solution of the one dimensional descent problem $x_t + \partial|x| \ni 0$, $x(0) = x_0$. We are going to generalize this simple result for function spaces.

We consider the measure space $(\Omega, d\mu)$ where $\Omega \subset \mathbb{R}^n$ and μ is a σ-finite measure. On the space $L^2(\Omega, d\mu)$ we define the functional

$$\Phi(u) = \begin{cases} \int_\Omega |u| d\mu, & \text{where the integral exists} \\ \infty, & \text{else.} \end{cases}$$

We will often use the more suggestive notation $\Phi(u) = \|u\|_{L^1(\Omega, d\mu)}$.

REMARK 2. *In the case of complex valued functions, we identify the space $L^2(\Omega,\mathbb{C},d\mu)$ with the real space $[L^2(\Omega,\mathbb{R},d\mu)]^2$. So we have the scalar product $\langle u|\,v\rangle_{L^2(\Omega,\mathbb{C},d\mu)} = \int_\Omega \langle u(x)|\,v(x)\rangle_{\mathbb{R}^2}\,d\mu$.*

One can show, that $\|u\|_{L^1(\Omega,d\mu)}$ is proper, convex and lsc on $L^2(\Omega,d\mu)$ so that the descent along $\partial\|u\|_{L^1(\Omega,d\mu)}$ has a solution. For an explicit solution of this descent, we have to know the subdifferential of Φ. With the use of the Fenchel-Legendre-Transform one can show, that

$$\partial\Phi(u) = \operatorname{sgn}(u)\cap L^2(\Omega,d\mu) \text{ with } \operatorname{sgn}(z) = \begin{cases} \left\{\frac{z}{|z|}\right\} & z\neq 0 \\ \{\zeta\in\mathbb{C} \mid |\zeta|\leq 1\} & z=0 \end{cases}$$

where $\operatorname{sgn}(u) = \{v\in L^2(\Omega,d\mu) \mid v(x)\in \operatorname{sgn} u(x) \text{ for almost every } x\}$.

Now we can prove the following

THEOREM 3.1. *Assume $u_0\in L^2(\Omega,d\mu)$. The shrinkage of u_0 with threshold t, which is given by*

$$u(t,x) = s_t(u_0(x)) := \operatorname{sgn}(u_0(x))[|u_0(x)|-t]_+,\ t\geq 0, x\in\Omega\ ,$$

is a solution of the subgradient descent problem

$$\frac{\partial u}{\partial t} + \partial\|u\|_{L^1(\Omega,d\mu)} \ni 0\ ,\ u(0) = u_0.$$

Proof. We replace the derivative with respect to t with backward differences in time. I. e. we choose a time step Δt and write $u^n(x)\approx u(n\Delta t,x)$ for the approximation to the true solution at time $n\Delta t$. The time discrete problem reads as

$$\frac{u^n-u^{n-1}}{\Delta t} + \partial\Phi(u^n)\ni 0 \ \Leftrightarrow\ \Delta t\partial\Phi(u^n)\ni u^{n-1}-u^n\ .$$

So for a given u^{n-1} we see by the definition of the shrinkage function that $u^n(x) = s_{\Delta t}(u^{n-1}(x))$ is a solution. So we define the operator $(T_{\Delta t}u)(x) := s_{\Delta t}(u(x))$. A classical result from semi-group theory says, that

$$u(t,\cdot) := \lim_{\substack{n\to\infty \\ n\Delta t = t}} T^n_{\Delta t} v_0$$

converges to a solution of our descent equation.

We observe, that $n\Delta t = t$ implies $T^n_{\Delta t} = T_t$, i.e. $T^n_{\Delta t}$ is a constant sequence. Therefore it converges to $T_t u_0 = s_t(u_0) = u(t,\cdot)$. ☐

Now we consider Fourier shrinkage as in equation (1). The shrinkage is applied on the transformed functions, i. e. in the frequency domain. We analyze, whether we can construct a differential formulation for the Fourier shrinkage in the spatial domain itself. From now on we will use the symbol $\mathcal{F}$ for the Fourier transform.

As before we consider the functional $\Phi(u) = \int_\Omega |u|d\mu$ an define $\Psi(u) := \Phi(\mathcal{F}u)$. A simple calculation shows, that $\partial\Psi(u) = \mathcal{F}^{-1}\partial\Phi(\mathcal{F}u)$. Here we needed, that the Fourier transform is bijective, i.e. the range of $\mathcal{F}$ is the whole $L^2(\mathbb{R})$. Then we can calculate that if v is a solution of the descent problem in the frequency domain

$$\frac{\partial v}{\partial t} + \partial\Phi(v)\ni 0,\ v(0,\cdot) = \mathcal{F}u_0$$

for a given initial value u_0, the solution of the problem in the spatial domain, namely

$$\frac{\partial u}{\partial t} + \partial\Psi(u)\ni 0,\ u(0,\cdot) = u_0\ ,$$

is given by $u = \mathcal{F}^{-1}v$. The Fourier shrinkage of a one dimensional signal can be seen in figure 1

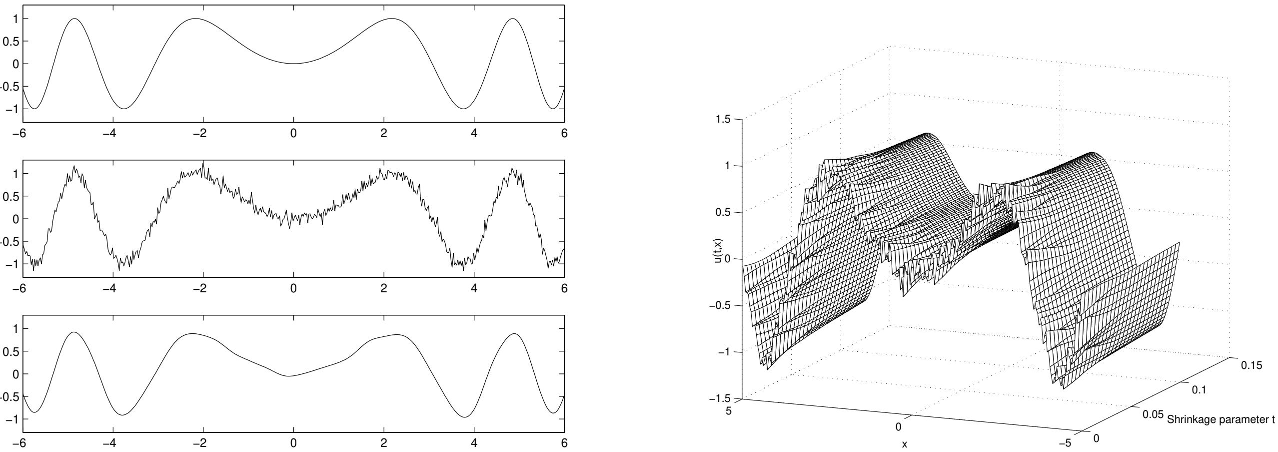

Figure 1. Left: The Fourier shrinkage of a noisy signal. From top to bottom: Original chirp signal. Signal disturbed with white noise of variance 0.1. The result of the Fourier shrinkage with shrinkage parameter $t = 0.1$. Right: The Fourier shrinkage of a noisy signal in a 3d-plot.

So we write more suggestive: The Fourier shrinkage of a function u_0 is a solution of the descent problem

$$\frac{\partial u}{\partial t} + \partial \|\mathcal{F}u\|_{L^1(\mathbb{R})} \ni 0\ ,\ u(0,\cdot) = u_0\ . \tag{3}$$

Encouraged by this result, we would like to find an analog description for the wavelet shrinkage in the spatial domain. However the wavelet transform is an isometry

$$L^2(\mathbb{R}) \to \operatorname{range} L_\psi \subset L^2(\mathbb{R}^2, \frac{dbda}{a^2})$$

but the range of the wavelet transform is a proper subspace of $L_2(\mathbb{R}^2, \frac{dbda}{a^2})$.

In particular $\operatorname{range} L_\psi$ is not invariant under shrinkage. Unfortunately, a subgradient descent in such a subspace is in general not a subgradient descent in the original Hilbert space (spatial domain) and vice versa. Nevertheless, we can analyze (3) in terms of the wavelet transform, i.e. we replace $\mathcal{F}$, resp. $\mathcal{F}^* = \mathcal{F}^{-1}$, by L_ψ, resp. L_ψ^* and obtain

$$\frac{\partial}{\partial t} u + L_\psi^* \partial \Phi_W(L_\psi u) \ni 0 \quad , \quad u(0,\cdot) = f \quad \text{where} \quad \Phi_W(v) = \int_{\mathbb{R}} \int_{\mathbb{R}} |v(a,b)| \frac{dbda}{a^2}\ .$$

Let us first analyze $L_\psi^* \partial \Phi_W(L_\psi u)$. The domain of definition, $\operatorname{dom} \partial\Phi_W$, is dense in the image space of the wavelet transform, however it intersects its range only in one point: Consider the subdifferential of a wavelet transformed function. The subdifferential $\partial\Phi_W(L_\psi f)$ is only non-empty, if $L_\psi f$ has integrable support (compare this to the calculation if the subgradient of $\|\cdot\|_{L^1}$.). However the support of a wavelet transformed function has a finite measure only if f is zero a.e., see.[18] Hence, $\operatorname{dom} L_\psi^* \partial\Phi_W(L_\psi) = \{0\}$.

Now we want to find a $g \in L^2(\mathbb{R})$ such that for a given $h \in L^2(\mathbb{R})$ the inclusion $h - g \in \Delta t L_\psi^* \partial \Phi_W(L_\psi g)$ is true. In fact, the only possibility is $g = 0$ since $L_\psi^* \partial\Phi_W(L_\psi g) = \emptyset$ for all other g.
Hence for any $u_{\Delta t}^0 = u_0$ and $u_{\Delta t}^n - u_{\Delta t}^{n+1} \in \Delta t L_\psi^* \partial\Phi(L_\psi u_{\Delta t}^{n+1})$ we obtain the limit $\lim_{\substack{n\to\infty \\ n\Delta t = t}} u_{\Delta t}^n = 0$ (if it exists). This shows that this construction, which is analog to the construction of a solution in theorem 3.1, does not give anything related to the wavelet shrinkage. In figure 2 we illustrated the effect of continuous wavelet shrinkage.

We end this section with a summary of these results on differential formulations for continuous shrinkage methods in the spatial domain (Fourier shrinkage), resp. image domain (wavelet shrinkage).

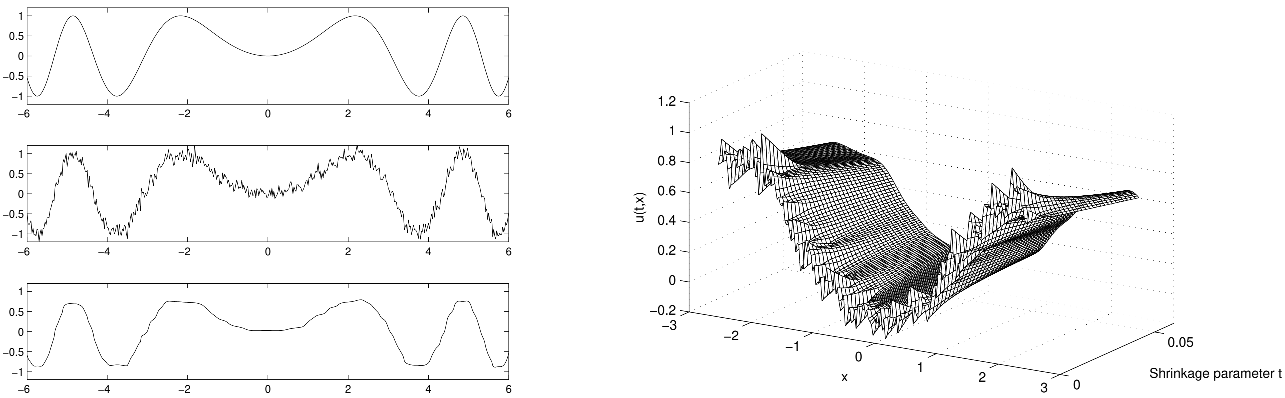

Figure 2. Left: The wavelet shrinkage with respect to the rational wavelet $\psi(x) = \frac{-2x}{(1+x^2)^2}$. The topmost picture shows the original signal for which white noise with variance 0.1 was added one picture below. Finally, the wavelet shrinkage of this signal with parameter $t = 0.028$ can be seen at the bottom. Right: The effect of the wavelet shrinkage of a noisy signal on multiple scales.

THEOREM 3.2. *Let $f \in L^2(\mathbb{R})$. Then a solution of the descent problem in the spatial domain*

$$\frac{\partial}{\partial t}u + \partial\|\mathcal{F}u\|_{L^1(\mathbb{R})}, \quad u(0,\cdot) = f$$

is given by the Fourier shrinkage of f:

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\operatorname{sgn}(\mathcal{F}f(\omega))[|\mathcal{F}f(\omega)| - t]_+)\, e^{i\omega x}\ d\omega.$$

The descent problem in the wavelet domain

$$\frac{\partial}{\partial t}v + \partial\|v\|_{L^1(\mathbb{R}^2, \frac{dbda}{a^2})}, \quad v(0,\cdot) = L_\psi f$$

has a solution v for which, s.t. $u(t,\cdot) = L_\psi^* v(t,\cdot)$, is the continuous wavelet shrinkage

$$u(t,x) = \frac{1}{\sqrt{c_\psi}} \int_{\mathbb{R}} \int_{\mathbb{R}} (\operatorname{sgn}(L_\psi f(a,b))[|L_\psi f(a,b)| - t]_+)\, |a|^{-1/2} \psi\left(\frac{x-b}{a}\right)\ \frac{dbda}{a^2}\ .$$

REMARK 3. *Finally we have to remark that our results for Fourier shrinkage are also valid for higher dimensions. We have the same equivalence as in equation (3) for signals $u_0 \in L^2(\mathbb{R}^n)$. The Fourier shrinkage is a subgradient descent along $\|\mathcal{F}u\|_{L^1(\mathbb{R}^n)}$. In figure 3 we show the Fourier shrinkage of a noisy image. The shrinkage is carried out directly in the Fourier domain. We show the absolute value of the reconstructed image.*

The higher dimensional wavelet case is (as in one dimension) more involved and depends on the type of higher dimensional wavelet transform you choose.

4. APPLICATION TO MAGNETOMYOGRAPHIC DATA

In this section, we present some results of an application of wavelet shrinkage to magnetomyographic data. The data set stems from the neuroscience center of the University of Tuebingen. In principle, the main objective is to examine the feasibility of recording the spatial-temporal magnetomyographic activity from the pregnant uterus

Figure 3. The Fourier shrinkage of a noisy image. Left: The original image. Middle: The image with Gaussian white noise. Right: The Fourier shrinkage of the noisy image.

Figure 4. Left: 151 channel sensor device; right: human fetus position in the sensor array.

with the use of a newly developed 151-channel noninvasive device, known as the superconducting quantum interference device (SQUID) array for reproductive assessment, see figure 4.

Since all electrophysiologic phenomena are characterized by the flow of ion currents within the body they can be detected by a measurement of potentials inside or on the surface of the body. The physics of electromagnetism predicts that the flow of current will also result in a magnetic field. By contrast, magnetic field recordings are much less dependent on tissue conductivity and are detectable outside the boundary of the skin without making electrical contact with the body. Unlike electrical recordings, the magnetic recordings are independent of any kind of reference, thus ensuring that each sensor mainly records localized activity.

The measured temporal data f_k (per channel k) mainly consists of a superposition of uterine activities, heart beat of mother, heart beat of fetus, brain activity of fetus, and certain noise. Hence, the first step in the context of signal processing is to decompose the temporal signal and to separate the components. In the second step one aims at detecting the spatial information of each signal component.

Here we have adjusted non-linear wavelet filtering in order to extract uterine activities. The performance of the wavelet scheme is visualized in Figure 5 (for channel f_5). The non-linear filtered data $\tilde{f}_5$ only contain contraction activities. The resulting data sets $\tilde{f}_k$, $k = 1, \ldots, 151$, and a simplified 2-dim. device geometry are the basic ingredients for further investigations of the spatial-temporal direction of uterine contraction. The resulting sequence of uterine activities is visualized in Figure 6. More details can be found in.[19]

ACKNOWLEDGMENTS

The authors would like to thank H. Preissl (University of Tuebingen) for stimulating discussions in the context of magnetomyographic data processing and for providing the data. Gerd Teschke gratefully acknowledges support by DFG grant Te 354/1-2.

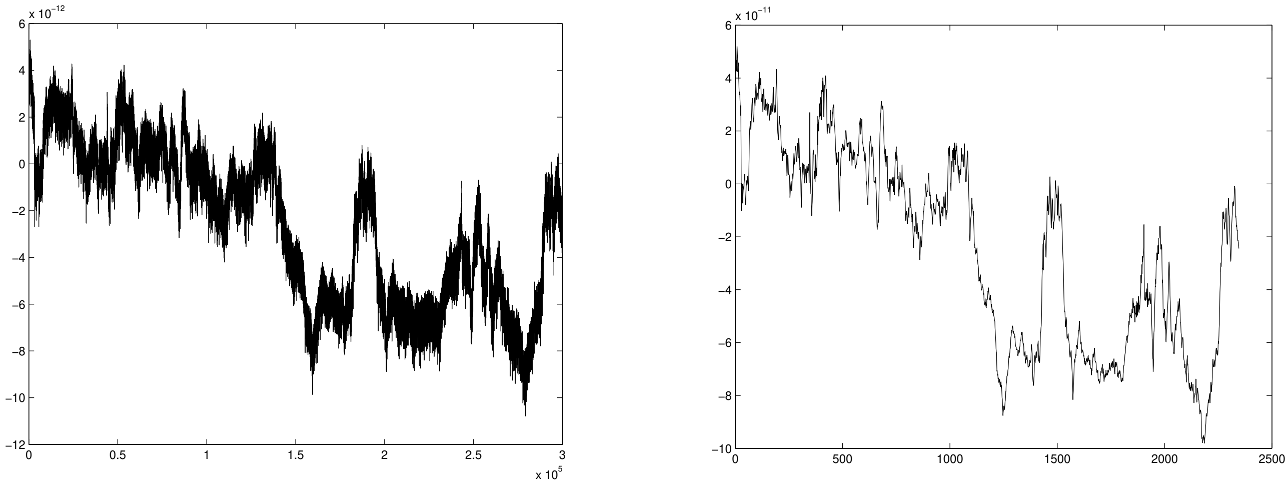

Figure 5. Left: original data f_5, right: non-linear wavelet filtered $\tilde{f}_5$.

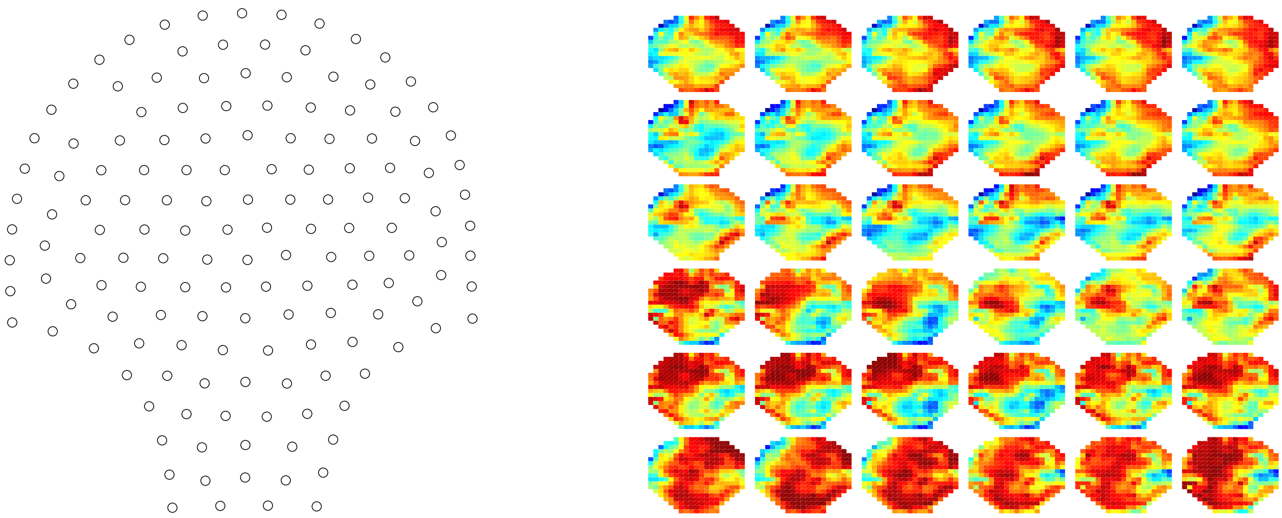

Figure 6. Left: simplified device geometry, right: temporal sequence of uterine contractions.

REFERENCES

1. K. Bredies, D. Lorenz, and P. Maass, "Mathematical concepts of multiscale smoothing." University of Bremen, Preprint Series of the DFG-Schwerpunktprogramm 1114, April 2003.
2. D. Donoho and I. Johnstone, "Minimax estimation via wavelet shrinkage," Tech. Rep. 402, Dept. of Statistics, Stanford University, 1992.
3. D. Donoho, I. Johnstone, G. Kerkyacharian, and D. Picard, "Density estimation by wavelet thresholding," tech. rep., Dept. of Statistics, Stanford University, 1993.
4. R. R. Coifman and D. L. Donoho, "Translation-invariant de-noising," in *Wavelets and Statistics*, A. Antoniadis and G. Oppenheim, eds., *Springer Lecture Notes in Statistics* **103**, pp. 125–150, Springer-Verlag, 1995.
5. H.-Y. Gao and A. G. Bruce, "WaveShrink with semisoft shrinkage," tech. rep., StatSci Division of MathSoft, Inc., 1995.
6. H.-Y. Gao, "Wavelet shrinkage denoising using the non-negative garrote," *Journal of Computational and Graphical Statistics* **7**(4), pp. 469–488, 1998.

7. A. Hyvärinen, P. Hoyer, and E. Oja, "Sparse code shrinkage: Denoising by nonlinear maximum likelihood estimation," *Advances in Neural Information Processing Systems* **11**, pp. 473–479, 1999.
8. A. Hyvärinen, P. Hoyer, and E. Oja, "Image denoising by sparse code shrinkage," in *Intelligent Signal Processing*, S. Haykin and B. Kosko, eds., IEEE Press, 2001.
9. L. Şendur and I. W. Selesnick, "Bivariate shrinkage functions for wavelet-based denoising exploiting interscale dependency," *IEEE Transactions on Signal Processing* **50**(11), pp. 2744–2756, 2002.
10. A. Antoniadis and J. Fan, "Regularization of wavelet approximations," *Journal of the American Statistical Association* **96**(455), pp. 939–967, 2001.
11. P. Moulin and J. Liu, "Analysis of multiresolution image denoising schemes using generalized gaussian and complexity priors," *IEEE Transactions on Information Theory* **45**(3), pp. 909–919, 1999.
12. M. A. T. Figueiredo and R. D. Nowak, "Wavelet-based image estimation: An empirical bayes approach using jefferey's noninformative prior," *IEEE Transactions on Image Processing* **10**(9), pp. 1322–1331, 2001.
13. G. Steidl and J. Weickert, "Relations between soft wavelet shrinkage and total variation denoising," in *Pattern Recognition*, L. V. Gool, ed., *Lecture Notes in Computer Science* **2449**, pp. 198–205, 2002.
14. D. A. Lorenz, "Methoden der Multiskalenglättung," Master's thesis, Universität Bremen, 2002.
15. A. Chambolle, R. A. DeVore, N.-Y. Lee, and B. J. Lucier, "Nonlinear wavelet image processing: Variational problems, compression and noiseremoval through wavelet shrinkage," *IEEE Transactions on Image Processing* **7**, pp. 319–335, 1998.
16. A. Chambolle and B. J. Lucier, "Interpreting translation-invariant wavelet shrinkage as a new image smoothing scale space," *IEEE Transactions on Image Processing* **10**, pp. 993–1000, 2001.
17. H. Brezis, *Opérateurs maximaux monotones et semi-groups de contractions dans les espaces de Hilbert*, vol. 5 of *North-Holland Mathematics Studies*, North-Holland Publishing Co., 1973.
18. E. Wilczok, "New uncertainty principles for the continuous Gabor transform and the continuous wavelet transform," *Documenta Mathematica* **5**, pp. 201–226, 2000.
19. G. Teschke and H. Verwold, "Waveletbasierte Analyse von MMG Daten," tech. rep., University Bremen, August 2002.