Interpolating Scaling Functions with Duals

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Abstract

This paper is concerned with the construction of smooth dual functions for a given family of interpolating scaling functions. The construction is based on a combination of the results in [2] and [11]. Several examples of dual functions are presented, including a continuously differentiable dual basis for the quincunx matrix.

Key Words: Interpolating scaling functions, Strang–Fix–conditions, biorthogonal wavelet bases, expanding scaling matrices, dual functions, Hölder regularity.

AMS Subject classification: 41A05, 42C40, 41A30, 41A63

1 Introduction

The construction of multivariate wavelets and scaling functions has been a field of increasing importance over the last years. A large variety of different construction principles has been published for orthogonal wavelets, biorthogonal wavelets, wavelets on spheres, scaling functions on general bounded and unbounded manifolds, scaling functions for specific operators (Radon transform, pseudo-differential operators, vaguelette bases) and many more.

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Wavelets are usually constructed by means of a so-called scaling function. In general, a function $\phi \in L_2(\mathbf{R}^d)$ is called a *scaling* function or a *refinable* function if it satisfies a *two–scale–relation*

$$\phi(x) = \sum_{k\in\mathbf{Z}^d} a_k \phi(Ax-k), \qquad \mathbf{a} = \{a_k\}_{k\in\mathbf{Z}^d} \in \ell_2(\mathbf{Z}^d), \tag{1.1}$$

where A is an *expanding* integer scaling matrix, i.e., all its eigenvalues have modulus larger than one.

Current interest centers around the construction of multivariate interpolating scaling functions ϕ, see e.g. [2, 3, 5, 6, 7, 8, 14], i.e. in addition to (1.1) one requires that ϕ is at least continuous and satisfies

$$\phi(k) = \delta_{0,k}, \qquad k \in \mathbf{Z}^d. \tag{1.2}$$

Interpolating scaling functions are needed for various applications e.g. CAGD or collocation methods for operator equations. These applications also require some smoothness of the scaling function. This problem has been solved satisfactory for ϕ itself, even for the notorious quincunx matrix.

The next step of the construction process asks to find a *dual* scaling function $\tilde{\phi}$ which satisfies

$$\langle \phi(\cdot), \tilde{\phi}(\cdot - k)\rangle = \delta_{0,k}, \quad k \in \mathbf{Z}^d. \tag{1.3}$$

However the best result so far for the quincunx matrix yields a dual scaling function $\tilde{\phi} \in C^\alpha$ with $\alpha = 0.3132$, see [11]. The aim of this paper is to construct duals for interpolating scaling functions which are continuously differentiable. In Section 5 a dual function $\tilde{\phi} \in C^\alpha$ for the quincunx matrix with $\alpha = 1.9528$ is constructed.

This result is based on a combination of three different techniques:

- construction of smooth interpolating multivariate scaling functions [2],
- construction of duals for interpolating scaling functions [11],
- estimating the regularity of scaling functions using the techniques of [15].

The construction of smooth dual functions is the cornerstone for further developments. Given such a dual function, there exist several ways to construct a *biorthogonal* wavelet basis, i.e., two sets $\{\psi_i\}_{i\in I}$ and $\{\tilde{\psi}_{i'}\}_{i'\in I}$ of functions satisfying

$$\langle |\det A|^{j/2}\psi_i(A^j\cdot -k), |\det A|^{j'/2}\tilde{\psi}_{i'}(A^{j'}\cdot -k')\rangle = \delta_{i,i'}\delta_{j,j'}\delta_{k,k'}, \tag{1.4}$$

see, e.g., [11] and [12] for details. Moreover, the existence of dual wavelets is essential for establishing characterizations of smoothness spaces such as Sobolev or Besov spaces. In fact, under certain regularity and approximation assumptions the existence of dual wavelets imply the equivalence of the Sobolev and Besov norms of a function to weighted sequence norms of its wavelet coefficients, see, e.g., [13] and [4] for details.

The construction of dual functions for interpolating scaling functions is a fairly recent research topic. First examples were obtained in [11]. This paper mainly deals with dual scaling functions for the classical box splines associated with the usual dyadic dilation matrix. Furthermore, some results concerning the quincunx matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ are included.

The results in [11] are derived by convolving a given interpolating scaling function with a suitable distribution. This distribution does not have any smoothness, i.e. this operation clearly diminishes the regularity of the resulting dual function $\tilde{\phi}$.

Therefore the whole construction only works satisfactory when the primal function ϕ is sufficiently smooth. Such a family of smooth interpolating scaling functions was constructed in [2].

Hence we apply the construction principle of [11] to the scaling functions constructed in [2], this leads to a new family of biorthogonal scaling functions for the quincunx matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ which has the advantage that the dual functions are much smoother when compared to the results in [11].

This paper is organized as follows. In Section 2, we briefly recall the basic setting of interpolating scaling functions. In Section 3, we explain the construction of [2] as far as it is needed for our purposes. Then, in Section 4, we recall the approach derived in [11]. Finally, in Section 5, we combine both approaches and present a detailed regularity analysis using the smoothness estimates of [15].

For later use, let us fix some notation. Let $q = |\det A|$. Furthermore, let $R = \{\rho_0, \ldots, \rho_{q-1}\}, R^T = \{\tilde{\rho}_0, \ldots, \tilde{\rho}_{q-1}\}$ denote complete sets of representatives of $\mathbf{Z}^d/A\mathbf{Z}^d$ and $\mathbf{Z}^d/B\mathbf{Z}^d$, $B = A^T$, respectively. Without loss of generality, we shall always assume that $\rho_0 = \tilde{\rho}_0 = 0$.

2 The Setting

In the sequel, we shall only consider compactly supported scaling functions, furthermore we shall always assume that supp $\mathbf{a} := \{k \in \mathbf{Z}^d \mid a_k \neq 0\}$ is finite. Computing the Fourier transform of both sides of (1.1) yields

$$\hat{\phi}(\omega) = \sum_{k \in \mathbf{Z}^d} \frac{1}{q} a_k e^{-i\langle k, B^{-1}\omega\rangle} \hat{\phi}(B^{-1}\omega). \tag{2.1}$$

By iterating (2.1) we obtain

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} a(e^{-iB^{-j}\omega}), \tag{2.2}$$

where the *symbol* $a(z)$ is defined by

$$a(z) := \frac{1}{q} \sum_{k \in \mathbf{Z}^d} a_k z^k. \tag{2.3}$$

Here we use the notation $z = z(\omega) = e^{-i\langle \cdot, \omega\rangle}$ and z^k is the short hand notation for $e^{-i\langle k, \omega\rangle}$. We will mainly use the z-notation in this paper, i.e. $a(1)$ refers to the value of the symbol at $\omega_1 = ...\omega_d = 0$. It will be stated explicitly, whenever we go back to the ω-notation.

All known procedures for constructing multivariate scaling functions start with a symbol $a(z)$, which by Equation (2.2) determines ϕ. Then the question arises which conditions on $a(z)$ guarantee that ϕ according to (2.2) is well–defined in $L_2(\mathbf{R}^d)$ and has some additional desirable properties such as sufficient smoothness. Moreover, for our purposes, we have to clarify how the interpolating property (1.2) can be guaranteed. The following two conditions are necessary:

(C1) $a(1) = 1$;

(C2) $\sum_{\tilde{\rho}\in R^T} a(\zeta_{\tilde{\rho}} e^{-iB^{-1}\omega}) = 1, \qquad \text{where} \quad \zeta_{\tilde{\rho}} := e^{-2\pi i B^{-1}\tilde{\rho}}.$

The following condition is not necessary, but it can be easily established in many cases and it is required for the construction of [11] as well as for the regularity estimates in Section 5. Moreover this condition already implies that the resulting scaling function is at least continuous:

(C3) $a(z) \geq 0$.

Usually, conditions (C1)–(C3) are the starting point for the construction of a suitable symbol and the related interpolatory scaling function. Nevertheless, we want to point out that they are not sufficient in general.

Several procedures are known for constructing interpolating scaling functions, however the true challenge asks for constructing smooth scaling functions. To this end, one often requires that the *Strang–Fix–conditions* of order N are satisfied, i.e.,

$$(C4) \qquad \left(\frac{\partial}{\partial\omega}\right)^l a(2\pi B^{-1}\tilde{\rho}) = 0 \qquad \text{for all} \quad |l| \leq N \quad \text{and all} \quad \tilde{\rho} \in R^T \backslash \{0\}.$$

This paper is concerned with the construction of pairs of biorthogonal functions $(\phi, \tilde{\phi})$ where ϕ is an interpolating scaling function and the dual scaling function $\tilde{\phi}$ satisfies (1.3). A necessary condition for the symbol $\tilde{a}$ of the dual scaling function $\tilde{\phi}$ in order to satisfy (1.3) is given by

$$1 = \sum_{\tilde{\rho}\in R^T} a(\zeta_{\tilde{\rho}} z)\overline{\tilde{a}(\zeta_{\tilde{\rho}} z)}. \tag{2.4}$$

Therefore the usual way to find a dual function for a given scaling function is to construct a symbol $\tilde{a}(z)$ satisfying (2.4) and to check that the corresponding refinable function exists in L_2 and is sufficiently regular. Indeed, we measure the success of a construction method for the dual function by the achievable Hölder regularity of $\tilde{\phi}$.

3 Smooth Interpolating Scaling Functions

As outlined in the introduction our search for smooth dual functions $\tilde{\phi}$ requires a smooth interpolating scaling function ϕ. The details on how to construct a suitable $\tilde{\phi}$, resp. $\tilde{a}$, for a given ϕ, resp. a are outlined in Section 4.

First of all we briefly recall the construction of interpolating scaling functions developed in [2]. It is based on Lagrange interpolation and can be interpreted as a generalization of the univariate approach derived in [10] to the multivariate situation.

We say that a symbol $a(z)$ satisfies the *Strang–Fix conditions* with respect to a set of polynomials Π, if ($D = \frac{\partial}{\partial\omega}$)

$$(p(D)a)(2\pi B^{-1}\tilde{\rho}) = 0 \quad \text{for all} \quad p \in \Pi,\ \tilde{\rho} \in R^T \backslash \{0\}. \tag{3.1}$$

For any subset $\mathcal{T} \subseteq \mathbf{Z}^d$, $\Pi_{\mathcal{T}}$ will always denote a finite–dimensional subspace of polynomials such that the Lagrange interpolation problem with respect to $\mathcal{T}$ is uniquely solvable. Under this hypothesis the following theorem holds.

Theorem 3.1 *Let $\mathcal{P}$ be a subspace of $\Pi_{\mathcal{T}}$ satisfying*

(1) If $p \in \mathcal{P}$, then $p\,(c(Ax+\rho)) \in \Pi_{\mathcal{T}}$ for $c \in \mathbf{C},\ \rho \in R$;

(2) $p(0) = 0$ for all $p \in \mathcal{P}$.

Then the symbol $a(\omega)$ defined by

$$a(\omega) = \frac{1}{q} + \frac{1}{q} \sum_{k \in \mathcal{T}} \sum_{\rho \in R\backslash\{0\}} p_k(-A^{-1}\rho) e^{-i\langle Ak+\rho,\omega\rangle} \tag{3.2}$$

satisfies (C1), (C2)*, and the Strang–Fix conditions* (3.1) *with respect to $\mathcal{P}$.*

Since Lagrange interpolation on general sets of nodes in $\mathbf{R}^d$ is far from understood, we restrict ourselves to very simple sets with additional symmetry. Let $\mathcal{T}$ consist of all lattice points in a cube in $\mathbf{R}^d$, i.e., for $N \in \mathbf{N}$ and $\beta \in \mathbf{Z}^d$ we set

$$\mathcal{T} = \mathcal{T}_{L,\beta} := \{k \in \mathbf{Z}^d :\ \beta_i \le k_i \le N + \beta_i, \quad i = 1, \ldots, d\} = (\beta + [0, N]^d) \cap \mathbf{Z}^d. \tag{3.3}$$

The Lagrange interpolation problem is always unisolvable on $\mathcal{T}$ by the polynomial subspace

$$\Pi_{\mathcal{T}} = \operatorname{span}\{x^k,\ k \in \mathbf{Z}^d, \|k\|_\infty \le N,\ k_i \ge 0, i = 1, \ldots, d\}\,. \tag{3.4}$$

The fundamental Lagrange interpolants are simply tensor products of the univariate Lagrange polynomials and can be written explicitly as

$$p_k(x) = \ell_{k_1}(x_1)\ell_{k_2}(x_2)\cdots\ell_{k_d}(x_d), \qquad \ell_{k_i}(x_i) := \prod_{n=a_i, n\ne k_i}^{L+a_i} \frac{x_i - n}{k_i - n}. \tag{3.5}$$

This leads to the following corollary.

Corollary 3.1 *Let $\mathcal{T}$ and $\Pi_{\mathcal{T}}$ be defined by* (3.3) *and* (3.4)*, respectively. Then $a(\omega)$ defined by* (3.2) *satisfies the Strang–Fix conditions with respect to $\Pi_{\mathcal{T}}$. In particular, the usual Strang–Fix conditions of order N are satisfied.*

It has been shown in [2] that under certain symmetry assumptions on the mask the resulting symbol is in fact real which is clearly necessary to ensure condition (C3). Moreover, in [2], this setting has been applied to the quincunx matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then $q = 2$ and a set of representatives is given by $\rho_0 = 0, \rho_1 = \binom{1}{0}$. Moreover, $-A^{-1}\binom{1}{0} = \binom{-1/2}{1/2}$ and $\mathcal{T}$ needs to be symmetric about $(-1/2, 1/2)$. This is the case for $\mathcal{T} = [-L, L-1] \times [-L+1, L] \cap \mathbf{Z}^2$. Let ℓ_n denote the basic Lagrange interpolation polynomial for $n \in \{-L, -L+1, .., L-1\}$. With

$$q_L(x) := \sum_{n=-L}^{L-1} \ell_n(-1/2) e^{-inx}, \tag{3.6}$$

we obtain for $a(\omega)$ corresponding to (3.2)

$$a(\omega) = \frac{1}{2} + \frac{1}{2} e^{-i(\omega_1+\omega_2)/2} q_L(\omega_1+\omega_2) e^{-i(\omega_2-\omega_1)/2} q_L(\omega_2-\omega_1). \tag{3.7}$$

By construction, this symbol satisfies (C1) and (C2). Moreover, it has been shown that for any L condition (C3) is also satisfied and that the symbol indeed gives rise to an interpolating scaling function.

As an example, for $L = 2$ the nonvanishing coefficients can be computed as follows.

$$\begin{aligned}
a_{(0,0)} &= \frac{1}{2}; \qquad (3.8)\\
a_{(1,0)} &= a_{(0,1)} = a_{(-1,0)} = a_{(0,-1)} = \frac{81}{512};\\
a_{(3,0)} &= a_{(0,3)} = a_{(-3,0)} = a_{(0,-3)} = \frac{1}{512};\\
a_{(2,1)} &= a_{(1,2)} = a_{(-1,2)} = a_{(-2,1)} = a_{(-2,-1)} = a_{(-1,-2)} = a_{(1,-2)} = a_{(2,-1)} = -\frac{9}{512}.
\end{aligned}$$

4 Construction of Dual Functions

In this section, we briefly recall the algorithm for constructing a dual basis for a given interpolating scaling function as developed in [11]. The main result in [11] is a lifting scheme, which allows to construct a second smoother interpolating function from a given one.

Defining

$$b_{\tilde{\rho}}(z) = a(\zeta_{\tilde{\rho}} z), \quad \tilde{\rho} \in R^T, \tag{4.1}$$

condition (C2) may equivalently be written as

$$1 = \sum_{\tilde{\rho} \in R^T} b_{\tilde{\rho}}(z). \tag{4.2}$$

Hence, for any integer K,

$$\left(\sum_{\tilde{\rho} \in R^T} b_{\tilde{\rho}}(z) \right)^{Kq} = \sum_{|\gamma|=qK} \left(C_{qK}^{\gamma} \prod_{\hat{\rho} \in R^T} b_{\hat{\rho}}^{\gamma_{\hat{\rho}}}(z) \right) = 1. \tag{4.3}$$

Here γ denotes a vector of dimension q, the coefficients of γ are indexed be $\tilde{\rho} \in R^T = \{\tilde{\rho}_0, \ldots, \tilde{\rho}_{q-1}\}$.

By using (4.3), the following theorem was established in [11].

Theorem 4.1 *Let $a(z)$ be a symbol satisfying (4.2) for a dilation matrix A with $q = |\det A|$. Define*

$$\begin{aligned} G_0 &:= \left\{ \gamma \in \mathbf{N}_0^q : \ |\gamma| = qK, \ \gamma_0 > K \ \text{and} \ \gamma_0 > \gamma_{\hat{\rho}}, \ \hat{\rho} \in R^T \backslash \{0\} \right\} \\ G_j &:= \left\{ \gamma \in \mathbf{N}_0^q : \ |\gamma| = qK, \ \gamma_0 > K \ \text{and} \ \gamma_0 \geq \gamma_{\hat{\rho}}, \ \hat{\rho} \in R^T \backslash \{0\}, \ \text{with exactly } j \text{ equalities} \right\}, \\ & \qquad j = 1, \ldots, q-2, \end{aligned}$$

and define

$$H_K := \sum_{j=0}^{q-2} \frac{1}{j+1} \left(\sum_{\gamma \in G_j} C_{qK}^{\gamma} a(z)^{\gamma_0 - 1} \prod_{\hat{\rho} \in R^T \backslash \{0\}} b_{\hat{\rho}}^{\gamma_{\hat{\rho}}}(z) \right) + C_{qK}^{(K,\ldots,K)} \prod_{\hat{\rho} \in R^T} b_{\hat{\rho}}^K(z),$$

where C_{qK}^{γ} are the multinomial coefficients. Then the symbol $a(z)H_K(z)$ also satisfies (4.2).

It can be checked that the symbol H_K can be factored as

$$H_K(z) = a(z)^K T_K(z) \tag{4.4}$$

for some suitable symbol $T_K(z)$. Consequently, the refinable function associated with $a(z)H_K(z)$ is obtained by convolving the original function $K-1$–times with itself followed by a convolution with some distribution. Since $a(z)H_K(z)$ satisfies (4.2), it is a candidate for a symbol corresponding to an interpolating scaling function. Indeed, the following corollary was established in [11].

Corollary 4.1 *Let $a(z)$ be the symbol of a continuous compactly supported interpolating refinable function and assume that $a(z)$ satisfies* (C3). *If the refinable function corresponding to $a(z)H_K(z)$ is continuous, then it is interpolating.*

This approach can now be used to construct dual functions for the given interpolating scaling function ϕ. Indeed, by recalling the necessary condition (2.4), we see that by Theorem 4.1

$$\tilde{a}(z) := \overline{H_K(z)} = \overline{a(z)^K T_K(z)} \tag{4.5}$$

is a natural candidate for a symbol associated with a dual function. The following corollary is again taken from [11].

Corollary 4.2 *If the refinable function corresponding to the mask H_K is in $L_2(\mathbf{R}^d)$, then it is stable and dual to ϕ.*

5 Smooth Dual Pairs on the Quincunx Grid

In this section, we want to employ the algorithm described in Section 4 to construct smooth dual pairs for the quincunx matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Corollary 4.2 tells us how to proceed:

- Find a continuous interpolating refinable function ϕ;
- Compute H_K according to Theorem 4.1;
- Check that the corresponding refinable function is contained in $L_2(\mathbf{R}^d)$.

Clearly the last part is the most nontrivial step. Moreover, it is desirable to find dual functions which are as smooth as possible. We are therefore faced with the problem of estimating the regularity of a refinable function by only using the refinement mask. This problem has attracted several people in the last few years, see, e.g., [1, 9, 14, 15]. Let us briefly recall the basic ideas. We want to find

$$\alpha^* := \sup\{\alpha : \ \phi \in C^\alpha\}.$$

It is well–known that $\alpha^* \geq \kappa_{\sup}$, where $\kappa_{\sup}$ is defined by

$$\kappa_{\sup} := \sup\{\kappa : \ \int_{\mathbf{R}^d} (1+|\omega|)^\kappa |\hat{\phi}(\omega)| d\omega < \infty\}. \tag{5.1}$$

Our aim is to estimate $\kappa_{\sup}$ from below. One typical result in this direction reads as follows.

Theorem 5.1 *For an integer N, let*

$$V_N := \{v \in \ell_0(\mathbf{Z}^d) \ : \ \sum_{k\in\mathbf{Z}^d} p(k)v_k = 0, \quad \text{for all } p \in \Pi_N\},$$

where Π_N denotes the polynomials of total degree N. Assume that A is a dilation matrix with a complete set of orthonormal eigenvectors, let $|\lambda_{max}|$ denote the eigenvalue of A with the largest modulus. Let Ω denote a subset of $\mathbf{Z}^d$ s.t. supp $\mathbf{a} \subseteq \Omega$ and V_N is invariant under the matrix

$$\mathcal{H} := [q a_{Ak-l}]_{k,l\in\Omega}.$$

Assume that the symbol $a(z)$ according to (2.3) is non–negative and satisfies Strang–Fix conditions of order N. Let ϱ be the spectral radius of $\mathcal{H}|_{V_N}$. Then the exponent $\kappa_{\sup}$ satisfies

$$\kappa_{\sup} \geq -\frac{\log(\varrho)}{\log(|\lambda_{\max}|)}. \tag{5.2}$$

As already stressed in Section 4, the approach in [11] actually consists of a convolution of the starting interpolating function ϕ with itself followed by a convolution with a distribution. This distribution may be ugly so that it may diminish the regularity of the resulting function significantly. Therefore the method in [11] will only perform satisfactory for a sufficienly smooth starting mask.

Hence we combine this construction procedure with the approach in [2], which produces interpolating functions with a small mask but with a high order of Strang–Fix conditions. Since the Strang–Fix conditions serve as indicators for some smoothness, there is good reason to expect that the resulting refinable functions are quite regular. Indeed, by using Theorem 5.1 we obtained for $L = 2$ and $L = 3$, respectively

$$\phi_2 \in C^\alpha \quad \text{for all } \alpha < 1.5156 \qquad \text{and} \qquad \phi_3 \in C^\alpha \quad \text{for all } \alpha < 2.3035. \tag{5.3}$$

Therefore we decided to use these functions as starting points. The next step is to compute the symbols H_K. For the quincunx matrix, we clearly have $q = 2$ and the first four symbols can be computed explicitly, for the definition of b_0 and b_1 see (4.1):

$$\begin{aligned} H_1 &= b_0(1 + 2b_1), \qquad (5.4)\\ H_2 &= b_0^2(b_0 + 4b_1 + 6b_1^2),\\ H_3 &= b_0^3(b_0^2 + 6b_0b_1 + 15b_1^2 + 20b_1^3),\\ H_4 &= b_0^4(b_0^7 + 8b_0^6b_1 + 28b_0^5b_1^2 + 56b_0^4b_1^3 + 70b_0^4b_1^4). \end{aligned}$$

For details, we refer again to [11]. Given $a(z)$, the corresponding symbols $H_1, \ldots, H_4$ can be computed by symbolic software such as `MAPLE`.

As an example, for $L = 2$ and $K = 1$ we obtain a mask with 65 non-zero coefficients:

$H_{1,(-6,0)} = -1/65536;$	$H_{1,(-5,-1)} = 9/32768;$	$H_{1,(-5,1)} = 9/32768;$
$H_{1,(-4,-2)} = -63/65536;$	$H_{1,(-4,0)} = -81/16384;$	$H_{1,(-4,2)} = -63/65536;$
$H_{1(-3,-3)} = -41/16384;$	$H_{1,(-3,-1)} = 567/32768;$	$H_{1,(-3,0)} = 1/256;$
$H_{1,(-3,1)} = 567/32768;$	$H_{1,(-3,3)} = -41/16384$	$H_{1,(-2-4)} = -63/65536;$
$H_{1,(-2,-2)} = 369/8192;$	$H_{1,(-2,-1)} = -9/256;$	$H_{1,(-2,0)} = -3969/65536;$
$H_{1,(-2,1)} = -9/256;$	$H_{1,(-2,2)} = 369/8192;$	$H_{1,(-2,4)} = -63/65536;$
$H_{1,(-1,-5)} = 9/32768;$	$H_{1,(-1-3)} = 567/32768;$	$H_{1,(-1,-2)} = -9/256;$
$H_{1,(1,-1)} = -2583/16384;$	$H_{1,(-1,0)} = 81/256;$	$H_{1,(-1,1)} = -2583/16384;$
$H_{1,(-1,2)} = -9/256;$	$H_{1,(-1,3)} = 567/32768;$	$H_{1,(-15)} = 9/32768;$
$H_{1,(0,-6)} = -1/65536;$	$H_{1,(0,-4)} = -81/16384;$	$H_{1,(0,-3)} = 1/256;$
$H_{1,(0,-2)} = -3969/65536;$	$H_{1,(0,-1)} = 81/256;$	$H_{1,(0,0)} = 6511/4096;$
$H_{1,(0,1)} = 81/256;$	$H_{1,(0,2)} = -3969/65536;$	$H_{1,(0,3)} = 1/256;$
$H_{1,(0,4)} = -81/16384;$	$H_{1,(0,6)} = -1/65536;$	$H_{1,(1,-5)} = 9/32768;$
$H_{1,(1,-3)} = 567/32768;$	$H_{1,(1,-2)} = -9/256;$	$H_{1,(1,-1)} = -2583/16384;$
$H_{1,(1,0)} = 81/256;$	$H_{1,(1,1)} = -2583/16384;$	$H_{1,(1,2)} = -9/256;$
$H_{1,(1,3)} = 567/32768;$	$H_{1,(1,5)} = 9/32768;$	$H_{1,(2,-4)} = -63/65536;$
$H_{1,(2,-2)} = 369/8192;$	$H_{1,(2,-1)} = -9/256;$	$H_{1,(2,0)} = -3969/65536;$
$H_{1,(2,1)} = -9/256;$	$H_{1,(2,2)} = 369/8192;$	$H_{1,(2,4)} = -63/65536;$
$H_{1,(3,-3)} = -41/16384;$	$H_{1,(3,-1)} = 567/32768;$	$H_{1,(3,0)} = 1/256;$
$H_{1,(3,1)} = 567/32768;$	$H_{1,(3,3)} = -41/16384;$	$H_{1,(4,-2)} = -63/65536;$
$H_{1,(4,0)} = -81/16384;$	$H_{1,(4,2)} = -63/65536;$	$H_{1,(5,-1)} = 9/32768;$
$H_{1,(5,1)} = 9/32768;$	$H_{1,(6,0)} = -1/65536.$	

(5.5)

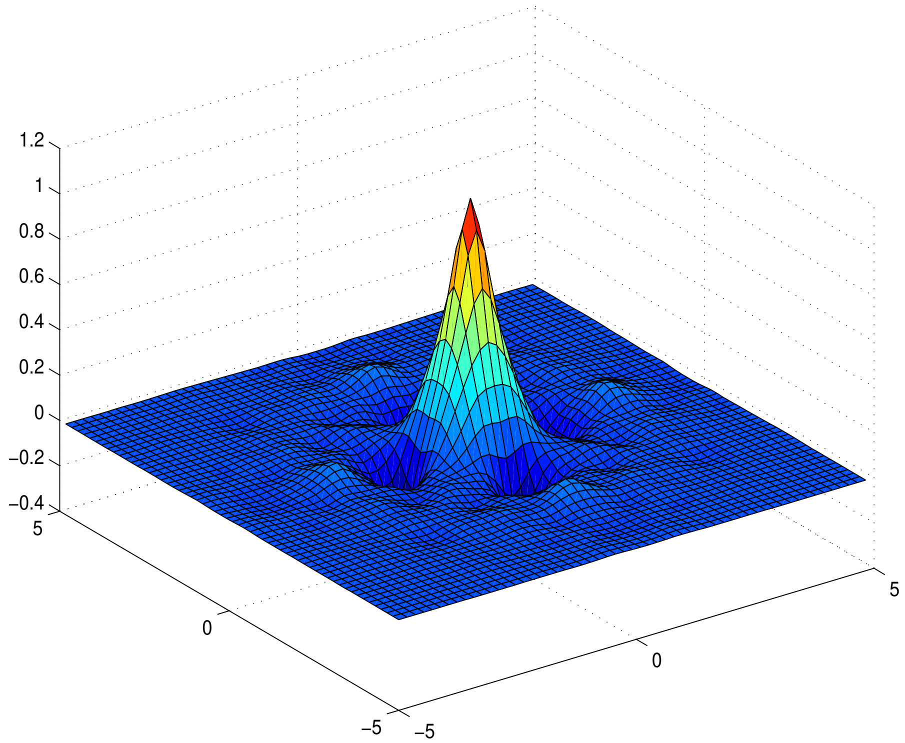

Figure 1: Visualization of the dual function for $L = 3,\ K = 2$, this function satisfies $\tilde{\phi} \in C^{\alpha}(\mathbf{R}^2)$ for $\alpha = 1.9528$.

We used Theorem 5.1 to estimate the regularity of the resulting refinable functions. The results are displayed in the following table.

L	K	κ_{sup}
2	1	-0.497
2	2	0.729
2	3	1.803
3	1	0.204
3	2	1.952

We see, that the regularity of the dual functions grows rapidly as K increases. For $L = 2,\ K = 1$ we do not get an L_2–function, but already the function with respect to $L = 2,\ K = 2$ is smoother than the smoothest one constructed in [11] which was contained in $C^{0.313226}$.

For $L = 2,\ K = 3$ the dual function is continuously differentiable. To our knowledge, examples for the quincunx matrix with these properties have not been constructed before. For $L = 3,\ K = 2$ the dual function is almost contained in C^2. It seems very likely that enlarging the values of N and K will produce even higher orders of regularity. However the computations become too time consuming, already the presented examples lead to eigenvalue problems for matrices with dimension $> 4 * 10^3$. This could only be handled with reasonable computer time by employing sparse matrix techniques.

References

[1] A. Cohen, K. Gröchenig, and L. Villemoes, Regularity of multivariate refinable functions, *Constr. Approx.* **15** (1999), 241–255.

[2] S. Dahlke, K. Gröchenig, and P. Maass, A new approach to interpolating scaling functions, *Appl. Anal.* **72(3–4)** (1999), 485–500.

[3] S. Dahlke and P. Maass, Interpolating refinable functions and wavelets for general scalings, *Numer. Funct. Anal. Optim.* **18(5&6)** (1997), 521–539.

[4] W. Dahmen, Stability of multiscale transformations, *J. Fourier Anal. Appl* **4** (1996), 341–362.

[5] J. Derado, Multivariate refinable interpolating functions, Appl. Comput. Harmon. Anal. **7** (1999).

[6] G. Deslauriers, J. Dubois, and S. Dubuc, Multidimensional iterative interpolation, *Can. J. Math.* **43** (1991), 297–312.

[7] G. Deslauriers and S. Dubuc, Interpolation dyadique, in: *Fractals, Dimensions non Entières et Applications*, (G. Cherbit, Ed.), Masson, Paris, 1987, pp. 44–45.

[8] S. Dubuc, Interpolation through an iterative scheme, *J. Math. Anal. Appl.* **114** (1986), 185–204.

[9] T. Eirola, Sobolev characterization of solutions of dilation equations, *SIAM J. Math. Anal.* **23** (1992), 1015–1030.

[10] M. Holschneider, "Wavelets: An Analysis Tool", Oxford Univ. Press, Oxford, 1995.

[11] H. Ji, S.D. Riemenschneider, and Z. Shen, Multivariate compactly supported fundamental refinable functions, dual and biorthogonal wavelets, to appear in: *Studies in Applied Math.*

[12] R.Q. Jia and C.A. Micchelli, Using the reefinement equation for the construction of Pre-wavelets II: Powers of two, in: "Curves and Surfaces", (P. J. Laurent, A. Le Méhauté and L. L. Schumaker, Eds.), Academic Press, New York, (1991).

[13] Y. Meyer, "Wavelets and Operators", Cambridge Studies in Advanced Mathematics, **vol. 37**, Cambridge, 1992.

[14] S.D. Riemenschneider and Z. Shen, Multidimensional interpolatory subdivision schemes, *SIAM J. Numer. Anal.* **34** (1997), 2357–2381.

[15] L. Villemoes, Wavelet analysis of refinement equations, *SIAM J. Math. Anal.* **25** (1994), 1433–1460.